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by Charles Libove, Dalpat Panchal, and Frank Dunn

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SYRACUSE UNIVERSITY RESEARCH INSTITUTE
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SUMMARY

A plane-stress analysis is presented for an isotropic or orthotropic elastic rectangular plate bounded by four edge stiffeners and subjected to prescribed temperature distributions and boundary loads. The stiffeners are assumed to have zero bending stiffness but finite extensional stiffness. They may be either uniform or tapered in such a way as to develop a constant (or non-constant but prescribed) cross-sectional stress. In the latter case the required form of taper is determined along with the stress.

The solution is by means of Fourier series. Its convergence and feasibility are tested by means of two numerical examples, one a thermal-stress problem, the other a "shear-lag" problem.

INTRODUCTION

A plane-stress analysis, by means of Fourier series, is presented for an elastic rectangular plate bounded by four edge stiffeners and subjected to prescribed loads and temperature distributions. The plate may be isotropic or orthotropic, with elastic constants that are independent of position and, if orthotropic, with axes of elastic symmetry parallel to the edges. The four edge stiffeners are assumed to have zero bending stiffness but finite extensional stiffness and to be integrally attached to the plate along their originally straight centroidal axes. The stiffeners are either uniform or tapered in such a way as to have constant (or non-constant but prescribed) cross-sectional stress; in the latter case the required form of taper of the stiffeners is determined along with the stresses.

More detailed descriptions of the structure follow, along with the results of the analysis. The symbols used are compiled and defined in appendix A. The details of the analysis, not required for the understanding and use of the results, are given in appendixes B and C.

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DETAILED DESCRIPTION OF STRUCTURE

Geometry and coordinate system. - The plate and stiffener combination is shown schematically in figure 1. The plate has a length of a and a width of b. Any point in the plate is identified by its coordinates x and y in a Cartesian reference frame whose origin is at one corner of the plate and whose axes coincide with two adjacent edges, as shown in the figure. The cross-sectional areas of the stiffeners are denoted by $A_1(y)$, $A_2(y)$, $A_3(x)$, and $A_4(x)$ for the stiffeners located at $x = 0$, $x = a$, $y = 0$, and $y = b$ respectively; however it is to be understood that these functions either are constants or have whatever form is necessary to produce constant (or non-constant but prescribed) stress in each stiffener.

Loading. - The assumed loading is also shown in figure 1. It consists of forces $P_1(0)$, $P_1(b)$, etc. applied to the centroids of the end cross sections of the stiffeners and distributed tensions $N_1(y)$, $N_2(y)$, $N_3(x)$, $N_4(x)$ and shear flows $q_1(y)$, $q_2(y)$, $q_3(x)$, $q_4(x)$ applied along the outside edges of the stiffeners. The distributed tensions and shear flows have dimensions of force per unit length. The loading as a whole is assumed to constitute an equilibrium system. Because the stiffeners have negligible bending stiffness, the distributed tensions are transmitted directly through them into the edges of the plate, however the shear flows are not transmitted unchanged to the edges of the plate.

Thermal strains. - The temperature distribution and hence the thermal strains are assumed to be known throughout the stiffeners and plate. The thermal-strain notation is indicated in figure 2 and is as follows: The thermal strains in the stiffeners are denoted by $e_1(y)$, $e_2(y)$, $e_3(x)$, and $e_4(x)$ for the stiffeners whose locations are $x=0$, $x=a$, $y=0$, $y=b$ respectively; the plate thermal strains are $e_x(x,y)$ and $e_y(x,y)$ in the x and y-directions respectively. All of these strains are assumed to be measured relative to some datum temperature distribution for which the structure is known to be stress-free. Note that there is no thermal shear strain relative to the x-and y-axes since these axes are parallel to the directions of elastic symmetry.

Stress-strain relations and elastic constants. - Figure 3 indicates the notation employed for the internal forces in the stiffeners and plate. $P_1(y)$ $P_2(y)$, $P_3(x)$, $P_4(x)$ denote the cross-sectional tensions and $\epsilon_1(y)$, $\epsilon_2(y)$, $\epsilon_3(x)$, $\epsilon_4(x)$ the total strains (thermal plus elastic) in the stiffeners located at $x = 0$, $x = a$, $y = 0$, $y = b$ respectively. The plate stress resultants (force per unit length) are represented by $N_x(x,y)$ and $N_y(x,y)$ for normal stress and $N_{xy}(x,y)$ for shear stress, as shown in figure 3. The corresponding total strains are symbolized by $\epsilon_x(x,y)$, $\epsilon_y(x,y)$, and $\gamma_{xy}(x,y)$.

With this notation established, the stress-strain relations for the stiffeners are assumed to have the form

$$\epsilon_i = e_i + \frac{P_i}{A_i E_i} \quad \left(i = 1,2,3,4\right) \tag{1}$$

with the Young's moduli E_1 and E_2 independent of y, E_3 and E_4 independent of x. The plate stress-strain relations are taken to be

$$\begin{aligned}
\epsilon_x &= e_x + C_1 N_x - C_3 N_y \\
\epsilon_y &= e_y + C_2 N_y - C_3 N_x \\
\gamma_{xy} &= C_4 N_{xy}
\end{aligned} \tag{2}$$

where the compliances C_1, C_2, C_3, and C_4 are independent of x and y. If the plate is homogeneous and isotropic, with thickness h, Young's modulus E, and Poisson's ratio ν, then

$$\begin{aligned}
C_1 = C_2 &= (Eh)^{-1} \\
C_3 &= \nu(Eh)^{-1} \\
C_4 &= 2(1+\nu)(Eh)^{-1}
\end{aligned}$$

SERIES EXPANSIONS FOR PRESCRIBED LOADS AND THERMAL STRAINS

The results of the present analysis, to be discussed shortly, consist of formulas for the stiffener and plate stresses in terms of the given loading and thermal-strain distribution. However, the loading and thermal strains do not appear explicitly in these formulas; it is rather the Fourier coefficients of these quantities that are required. In anticipation of this requirement, it is assumed that the given distributed loadings can be expressed in Fourier series of the following form, with known coefficients:

$$N_1(y) = \sum_{n=1}^{N} B'_n \sin (n\pi y/b)$$

$$N_2(y) = \sum_{n=1}^{N} B''_n \sin (n\pi y/b)$$

$$N_3(x) = \sum_{m=1}^{M} B'''_m \sin (m\pi x/a) \tag{3}$$

$$N_4(x) = \sum_{m=1}^{M} B''''_m \sin (m\pi x/a)$$

$$q_1(y) = \sum_{n=0}^{N} Q'_n \cos (n\pi y/b)$$

$$q_2(y) = \sum_{n=0}^{N} Q''_n \cos (n\pi y/b)$$

$$q_3(x) = \sum_{m=0}^{M} Q'''_m \cos (m\pi x/a) \tag{4}$$

$$q_4(x) = \sum_{m=0}^{M} Q''''_m \cos (m\pi x/a)$$

Similarly, if there are any discontinuities in thermal strain between the stiffeners and the plate edges, these are assumed to be known in the form of the following Fourier series:

$$
\left.
\begin{aligned}
e_1(y) - e_y(0,y) &= \sum_{n=1}^{N} T_n' \sin (n\pi y/b) \quad &\text{for} \quad 0 < y < b \\
e_2(y) - e_y(a,y) &= \sum_{n=1}^{N} T_n'' \sin (n\pi y/b) \quad &\text{for} \quad 0 < y < b \\
e_3(x) - e_x(x,0) &= \sum_{m=1}^{M} T_m''' \sin (m\pi x/a) \quad &\text{for} \quad 0 < x < a \\
e_4(x) - e_x(x,b) &= \sum_{m=1}^{M} T_m'''' \sin (m\pi x/a) \quad &\text{for} \quad 0 < x < a
\end{aligned}
\right\} \tag{5}
$$

Finally, $\partial^2 e_y/\partial x^2 + \partial^2 e_x/\partial y^2$ is assumed to be representable by the following series in the open region $0 < x < a$, $0 < y < b$:

$$
\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2} = \sum_{m=1}^{M} \sum_{n=1}^{N} T_{mn} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \tag{6}
$$

Finite upper limits M and N are shown for the summation indexes in equations (3) to (6) in expectation of the fact that it will normally be necessary to use truncated rather than infinite series for practical computational reasons.

In the analysis of the constant-area-stiffeners case in appendix B the Fourier coefficients of the loading and thermal strains will appear in certain groups. These groups are designated by the letters R_n', R_n'', R_m''', R_m'''' for convenience and are defined by equations (B66) and (B67).

The Fourier coefficients appearing in equations (3) to (6) and required for the evaluation of the R_n', R_n'', R_m''', R_m'''' can be determined from the usual definitions. For example,

$$
\left.
\begin{aligned}
B_n' &= \frac{2}{a} \int_0^b N_1(y) \sin \frac{n\pi y}{b} \, dy \\
Q_n' &= \frac{2 - \delta_{n0}}{b} \int_0^b q_1(y) \cos \frac{n\pi y}{b} \, dy \\
T_n' &= \frac{2}{b} \int_0^b [e_1(y) - e_y(0,y)] \sin \frac{n\pi y}{b} \, dy
\end{aligned}
\right\} \tag{7}
$$

$$
T_{mn} = \frac{4}{ab} \int_0^b \int_0^a \left(\frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2}\right) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dx dy \tag{8}
$$

where δ_{n0} is Kronecker's delta. Integration by parts in the last equation gives the following alternate formula which permits T_{mn} to be evaluated from the first derivatives of e_y and e_x instead of the second derivatives:

$$T_{mn} = -\frac{m\pi}{a}\frac{4}{ab}\int_0^b\int_0^a \frac{\partial e_y}{\partial x}\cos\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$-\frac{n\pi}{b}\frac{4}{ab}\int_0^a\int_0^b \frac{\partial e_x}{\partial y}\cos\frac{n\pi y}{b}\sin\frac{m\pi x}{a}\,dy\,dx \tag{9}$$

Equation (9) may be used for discontinuous e_y or e_x provided that $\partial e_y/\partial x$ and $\partial e_x/\partial y$ are regarded to be infinite, in the manner of the Dirac delta function, at the loci of points of discontinuity. <u>If e_y and e_x are continuous</u> in the region $0 \leq x \leq a$, $0 \leq y \leq b$, further integration by parts gives

$$T_{mn} = -\frac{m\pi}{a}\frac{4}{ab}\int_0^b \left[e_y(a,y)\cos m\pi - e_y(0,y)\right]\sin\frac{n\pi y}{b}\,dy$$

$$-\left(\frac{m\pi}{a}\right)^2\frac{4}{ab}\int_0^b\int_0^a e_y(x,y)\sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dx\,dy$$

$$-\frac{n\pi}{b}\frac{4}{ab}\int_0^a \left[e_x(x,b)\cos n\pi - e_x(x,0)\right]\sin\frac{m\pi x}{a}\,dx$$

$$-\left(\frac{n\pi}{b}\right)^2\frac{4}{ab}\int_0^a\int_0^b e_x(x,y)\sin\frac{n\pi y}{b}\sin\frac{m\pi x}{a}\,dy\,dx \tag{10}$$

RESULTS FOR CONSTANT-AREA STIFFENERS

<u>Series for the plate and stiffener stresses</u>.- The analysis in appendix B gives the plate stress resultants and stiffener forces in the form of series. Equations (B21) to (B27) give the plate stress resultants N_y, N_x, and N_{xy}, equations (B17) the stiffener forces. It should be noted that the series given for N_y in the interior of the plate, equation (B21), is not

valid along the edges; separate series, equations (B22) and (B23), are given for evaluating N_y along the edges $x = 0$ and $x = a$; along the other two edges, $y = 0$ and $y = b$, N_y is equal to $N_3(x)$ and $N_4(x)$ respectively. Analogous remarks apply to N_x. Similarly the series for the stiffener forces, equations (B17), are not valid at the ends of the stiffeners; this imposes no great hardship, however, since the stiffener end forces are known as part of the given loading.

Evaluation of series coefficients. - In order to use these series for numerical calculation of stresses, one must first evaluate the coefficients c_n', c_n'', g_m', g_m'', c_{mn}, g_{mn}, j_{mn}, s_n', s_n'', s_m''', and s_m'''' appearing in them. The first four groups of coefficients, namely c_n', c_n'', g_m', g_m'', are the key to all the others, so their evaluation will be discussed first.

The c_n', c_n'', g_m', and g_m'' are defined by the system of equations (B61) to (B64) and can be determined by solving these 2N + 2M equations simultaneously for the $\bar{c}_n'$, $\bar{c}_n''$, $\bar{g}_m'$, $\bar{g}_m''$ and noting the definitions in equations (B65). As an alternative, equations (B70) may be solved simultaneously for the g_m' and g_m''; the c_n' and c_n'' are then obtained directly from equations (B68). This alternative is preferable because it requires the solution of only 2M simultaneous equations, regardless of how large a value is selected for N.

If the structure, loading and thermal strains are symmetrical about the centerline $y = b/2$, considerable simplification results. The c_n', c_n'', g_m', and g_m'' are then defined by equations (B78) and the simultaneous system of M + N + 1 equations (B61') to (B63'). As a preferable alternative to equations (B61') to (B63'), the M equations (B70') may be solved simultaneously for the g_m', after which the g_m'', c_n', and c_n'' are obtained directly from equations (B78) and (B68). In this alternative the size of N again does not influence the number of equations that have to be solved simultaneously.

If both $y = b/2$ and $x = a/2$ are axes of symmetry of the structure, loading, and thermal strains, then equations (B79), (B61"), (B63") may be used to obtain the c_n', c_n'', g_m', g_m'', where (B61") and (B63") represent $(M + N + 2)/2$ simultaneous equations. The quantities may also be determined from fewer (namely $(M + 1)/2$) simultaneous equations by using the system (B70") to solve for the g_m' and then obtaining the remaining coefficients directly from equations (B79) and (B61").

With the c_n', c_n'', g_m', g_m'' known, equations (B59), (B60), (B58), and (B56) will furnish the remaining coefficients directly.

Limiting case: large stiffener areas. - If the stiffener cross-sectional areas are assumed to approach infinity while maintaining constant ratios with each other*, the solution takes on a much simplified form characterized by the fact that it is no longer necessary to solve simultaneous equations. The quantities c_n', c_n'', g_m', and g_m'' are defined by equations (B80) directly on this limiting case.

In equations (B80) the coefficients of $R_n'/(A_1E_1)$ etc., are in error by terms of the order of $(a^3E_{11}A_1E_1)^{-1}$. A more accurate solution, in which these coefficients are correct to terms of the order of $(a^3E_{11}A_1E_1)^{-1}$, is represented by equations (B81) to (B84). This more accurate solution still retains some of the simplicity of equations (B80), in the sense that it too does not require the solution of simultaneous equations.

Numerical examples. - The foregoing results were used to obtain numerical stress data for two illustrative problems, one a thermal-stress problem involving non-uniform temperature distribution without any applied loads, the second a "shear-lag" problem involving the diffusion of loads from the stiffener ends into the plate. The two problems are shown schematically in figure 4.

In both problems the plate is square (b = a) and isotropic, with Young's modulus E, Poisson's ratio ν, and thickness h. The four stiffeners are assumed to be identical and to have the same Young's modulus as the plate. The symbol A will designate the common values of A_1, A_2, A_3, and A_4.

In the thermal stress problem (figure 4(a)), the stiffeners are at a uniform temperature, say zero, while the plate has a "pillow-shaped" temperature distribution of the form $\theta \sin(\pi x/a) \sin(\pi y/a)$; thus θ denotes the temperature rise of the plate center relative to the edges. The symbol α will denote the coefficient of thermal expansion of the materials.

In the shear-lag problem (Figure 4(b)), the temperature is uniform and the loading consists of identical tension loads of magnitude P applied to the end cross sections of the stiffeners.

In these problems the structure, loading, and thermal strains are symmetrical about both centerlines, namely x and y = a/2. Hence equations (B70") may be used for their analysis. In these equations R_n' and R_m''' take the following forms, according to equations (B66) and (B67), for the thermal-stress problem:

---

*More precisely, if the dimensionless quantities $(a^3E_{11}A_iE_i)^{-1}$, with i = 1, 2, 3, 4, are assumed to approach zero while maintaining constant ratios.

$$R'_n = -\sum_{m=1}^{M} K_{mn} \qquad\qquad R'''_m = -\sum_{n=1}^{N} K_{mn}$$

where

$$K_{mn} = mn\pi^2 T_{mn}/(a^2 E_{mn})$$

$$E_{mn} = (m^2 + n^2)^2 \pi^4/(a^4 Eh)$$

$$T_{mn} = \begin{cases} -2\alpha\theta\pi^2/a^2 & \text{when} \quad m = n = 1 \\ 0 \text{ otherwise} & \end{cases}$$

Thus

$$R'_n = \begin{cases} Eh\alpha\theta/2 & \text{for } n = 1 \\ 0 & \text{for } n \neq 1 \end{cases}$$

$$R'''_m = \begin{cases} Eh\alpha\theta/2 & \text{for } m = 1 \\ 0 & \text{for } m \neq 1 \end{cases} \tag{11}$$

For the shear-lag problem, on the other hand,

$$R'_n = R'''_m = 4P/a \tag{12}$$

for all (odd) n and m.

The results of the calculations are shown in the figures, starting with figure 5, through dimensionless plots of the plate stress resultants N_x, N_y, and N_{xy} as functions of x for fixed values of y, and the stiffener force $P_3(x)$. (In view of the symmetry which exists about the plate diagonals in these examples, as well as about the plate centerlines, the graphs of $P_2(y)$, $P_1(y)$, and $P_4(x)$ are all identical with the graph for $P_3(x)$.) The results shown are based on calculations with M = N = 39 . As a check on the convergence, the calculations were repeated with M = 39, N = 79; the results agreed with the previous ones to the extent that any differences would be indiscernible on the graphs shown. The dotted curves in some of the figures represent approximate solutions obtained by using equations (B80) and (B81) to (B84), which apply when the stiffener cross-sectional areas are large compared to the plate cross-sectional area. It is seen that the large-stiffener approximation is quite good for the plate stresses when the area-ratio parameter $4ah/(\pi^2 A)$ is 0.1 or smaller; it is of course exact when $4ah/(\pi^2 A) = 0$.

The thermal-stress results indicate finite shear stress in the corner of the plate for all values of the parameter $4ah/(\pi^2 A)$. However, the shear-lag problem exhibits infinite corner shear stress for the same values of the parameter $4ah/(\pi^2 A)$. In an actual structure, finite stiffener bending stiffness (in particular, rigid connections where the stiffeners meet at the corners) would lead to zero shear stress at the corner and therefore, in a neighborhood near the corner, to shear stresses which might be considerably different from those which the present analysis predicts.

Accepting the premise of zero bending stiffness of the stiffeners or hinged connections where the stiffeners meet, it is interesting to note that in the thermal-stress problem the maximum value of N_{xy} in the plate does not always occur at the corner (see, for example, figure 5c).

The case $4ah/(\pi^2 A) = 0$ represents the limiting case in which the sheet is infinitely thin by comparison with the stiffener cross-sectional dimensions. In this case the end loads applied to the stiffeners in the shear-lag problem should be transmitted unchanged throughout the lengths of the stiffeners. Thus in figure 6(a) the value of $P_3(x)/P$ should be 1.0 for all x/a, and $N_x A/(hP)$ should similarly be 1.0 for all x/a along the line $y/a=0.0$. The deviations from 1.0 shown in figure 6(a) are due to incomplete convergence associated with the use of finite values of M and N.

The calculations were performed on the IBM 7070 computer at the Syracuse University Computing Center and required a total of 50.5 minutes for all the results shown. The simultaneous equations (B70") were solved by the iterative procedure of reference 2. The plate stresses were computed at x/a and y/a intervals of 0.1 in order to plot the curves shown.

RESULTS FOR CONSTANT-STRESS (OR PRESCRIBED STRESS) STIFFENERS

For efficient design, it may be desired to taper the stiffeners so as to achieve constant cross-sectional stress along the length of each one. In appendix C a generalization of this condition is considered in which it is assumed that the tapers are such as to produce prescribed, but not necessarily constant, cross-sectional stress variations along the lengths of the stiffeners. The symbols $\sigma_1(y)$, $\sigma_2(y)$, $\sigma_3(x)$ and $\sigma_4(x)$ represent the prescribed stresses (positive for tension) in the stiffeners located at $x = 0$, $x = a$, $y = 0$, and $y = b$ respectively.

In this case it turns out once again to be unnecessary to solve simultaneous equations in order to compute the plate stresses. Equations

(B33), (B34), (B36), and (B37), used in conjunction with equations (C2), give the c'_n, c''_n, g'_m, and g''_m directly; equations (B60), (B58), and (B56) then furnish the j_{mn}, and equations (B21), (B24), and (B27) the plate stresses.

The lengthwise variations of stiffener cross-sectional area needed to produce the prescribed stiffener stresses can be determined from equations (C3) in conjunction with (C4) to (C7).

CONCLUDING REMARKS

A plane-stress analysis, based on Fourier series, has been presented for the stresses in a linearly elastic isotropic or orthotropic rectangular plate bounded by four edge stiffeners and subjected to prescribed temperature distributions and boundary loadings. The stiffeners are assumed to be either uniform or tapered in such a way as to develop constant (or nonconstant but prescribed) stresses. The convergence and feasibility of the analysis have been tested and found to be satisfactory in two numerical examples for the case of uniform stiffeners.

The present analysis differs from previous thermal-stress analyses of rectangular plates by the incorporation of edge stiffeners. It differs from previous "shear-lag" analyses by the avoidance of extreme assumptions regarding the plate stiffness or plate normal stresses in what is usually called the transverse direction.

However the present analysis is also characterized by the assumption of zero bending stiffness for the stiffeners and the absence of boundary conditions on displacement. The removal of these two limitations would, it is felt, be a worthwhile objective of further research. Regarding the first limitation in particular, the inclusion of finite stiffener bending stiffness and joint rigidity where the stiffeners meet appears to be necessary in order to obtain realistic estimates of shear stress in the neighborhoods of the corners.

APPENDIX A

SYMBOLS

Remarks: (i) The subscript 1,2,3 or 4 on a symbol for a stiffener-related quantity identifies the stiffener location as $x = 0$, $x = a$, $y = 0$, or $y = b$ respectively. (ii) The Fourier coefficients of known quantities (loads, thermal strains), and combinations of such coefficients, are generally represented by capital letters, while the Fourier coefficients of initially unknown quantities (e.g., internal stresses) are denoted by small letters.

a	plate dimension in x direction; see figure 1
a_{mn}	Fourier coefficients in series expansion for the stress function $F(x,y)$; see equation (B16)
$a_n', a_n'', a_m''', a_m''''$	Fourier coefficients in series expansions for $F(0,y)$, $F(a,y)$ $F(x,0)$, $F(x,b)$ respectively; see equations (B14)
$A_1(y), A_2(y), A_3(x), A_4(x)$	stiffener cross-sectional areas
A	common value of the above when all four stiffeners are identical and uniform
b	plate dimension in y direction; see figure 1
$B_n', B_n'', B_m''', B_m''''$	Fourier coefficients in series expansions for N_1, N_2, N_3, N_4 respectively; see equations (3) and (7)
c_{mn}	Fourier coefficients in series expansion for $N_y(x,y)$; see equation (B21)
c_n', c_n''	Fourier coefficients in series expansions for $N_y(0,y)$ and $N_y(a,y)$ respectively; see equation (B22)
$\bar{c}_n', \bar{c}_n''$	$c_n' C_2 n\pi/b$, $c_n'' C_2 n\pi/b$
C_1, C_2, C_3, C_4	plate compliances defined by equations (2)
e_{mn}	Fourier coefficients in series expansion for $\partial^4 F/\partial x^4$; see equation (B28)

$e_1(y)$, $e_2(y)$, $e_3(x)$, $e_4(x)$	Stiffener thermal strains; see figure 2
$e_x(x,y)$, $e_y(x,y)$	plate thermal strains; see figure 2
E_{mn}	$C_2(m\pi/a)^4+(C_4-2C_3)(m\pi/a)^2(n\pi/b)^2+C_1(n\pi/b)^4$
E_{11}	value of above with m = 1 and n = 1
E_1, E_2, E_3, E_4	Young's moduli of stiffeners
E	Young's modulus for stiffeners and isotropic plate when all have the same Young's modulus
F(x,y)	stress function for plate; see equation (B4)
g_{mn}	Fourier coefficients in series expansion for $N_x(x,y)$; see equation (B24)
g'_m, g''_m	Fourier coefficients in series expansions for $N_x(x,0)$ and $N_x(x,b)$ respectively; see equations (B25) and (B26)
$\bar{g}'_m$, $\bar{g}''_m$	$g'_m C_1 m\pi/a$, $g''_m C_1 m\pi/a$
h	thickness when plate is isotropic
i	1,2,3, or 4
i_{mn}	Fourier coefficients in series expansion $\partial^4 F/\partial y^4$; see equation (B29)
j_{mn}	Fourier coefficients in series expansion for N_{xy} ; see equation (B27); see equation (B58) for value of j_{00}
K_{mn}	combinations of known Fourier coefficients, defined by equation (B67)
L_{Mn}	known quantity defined by equation (B72)
m, n, p, q	summation indexes (integers)
M	upper limit on m, p, and q
n	summation index (integer)
N	upper limit on n

$N_1(y)$, $N_2(y)$, $N_3(x)$, $N_4(x)$	external running tensions, force per unit length; see figure 1
N_x, N_y, N_{xy}	plate stress-resultants, force per unit length; see figure 3
p	summation index (integer)
$P_1(y)$, $P_2(y)$, $P_3(x)$, $P_4(x)$	stiffener cross-sectional tensions; see figure 3
$P_1(0), P_1(b)$, $P_2(0)$, $P_2(b)$, $P_3(0)$, $P_3(a)$, $P_4(0)$, $P_4(a)$	stiffener end loads; see figure 1
P	common value of the above when all are equal; used in numerical example
q	summation index (integer)
$q_1(y)$, $q_2(y)$, $q_3(x)$, $q_4(x)$	external shear-flow loadings; see figure 1
Q_n', Q_n'', Q_m''', Q_m''''	Fourier coefficients in series expansions for q_1, q_2, q_3, q_4 respectively; see equations (4) and (7)
R_n', R_n'', R_m''', R_m''''	combinations of known Fourier coefficients, defined by equations (B66) and (B67)
s_n', s_n'', s_m''', s_m''''	Fourier coefficients in series expansions for the stiffener cross-sectional tensions; see equations (B17)
t_n', t_n'', t_m''', t_m''''	Fourier coefficients in series expansions for the derivatives of the stiffener cross-sectional tensions; see equations (B20) and (B42)
T_{mn}	Fourier coefficients in series expansion for $\partial^2 e_y/\partial x^2 + \partial^2 e_x/\partial y^2$; see equation (6), also equations (8), (9), and (10)
T_n', T_n'', T_m''', T_m''''	Fourier coefficients in series expansions for thermal-strain discontinuities between stiffeners and plate edges; see equations (5) and (7)
u, v	x and y components of displacements in plate

U'_{mMN}, U''_{mMN}	known quantities defined by equations (B71)
v	plate displacement component in y-direction
V_{Mmnp}	known quantity defined by equations (B76) and (B77)
W'_{MNmp}, W''_{MNmp}	known quantities defined by equations (B75)
x,y	Cartesian coordinates; see figure 1
x'	dummy variable representing x
X'_{Mn}, X''_{Mn}, X'''_{Nm}, X''''_{Nm}	known quantities defined by equations (B73)
y	Cartesian coordinate; see figure 1
y', y''	dummy variables representing y
Y'_{Mn}, Y''_{Nm}	known quantities defined by equations (B74)
Z'_{Mn}, Z'''_{Nm}	known quantities defined by equations (B73")
α	coefficient of thermal expansion of plate and stiffeners in numerical example
δ_{n0}, δ_{m0}, δ_{mp}	Kronecker's delta; unity when both subscripts are equal, zero otherwise
δ_{mpq}	1 for m, p both even and q odd; 1 for m, p both odd and q even; zero otherwise
Δ_n	known quantity defined by equation (B69)
$\epsilon_x(x,y)$, $\epsilon_y(x,y)$, $\gamma_{xy}(x,y)$	plate total strains
$\epsilon_1(y)$, $\epsilon_2(y)$, $\epsilon_3(x)$, $\epsilon_4(x)$	stiffener total strains
θ	temperature rise of plate center relative to the edges, used in numerical example
ν	Poisson's ratio when plate is isotropic
$\sigma_1(y)$, $\sigma_2(y)$, $\sigma_3(x)$, $\sigma_4(x)$	prescribed stiffener stresses, positive for tension
λ	$4ah/(\pi^2 A)$; area-ratio parameter used in presenting the results of the numerical examples.

APPENDIX B

ANALYSIS FOR THE CASE OF CONSTANT-AREA STIFFENERS

Basic equations. - With $u(x,y)$ and $v(x,y)$ denoting the x- and y-components of infinitesimal displacement, the strain-displacement relations for the plate are

$$\epsilon_x = \frac{\partial u}{\partial x} \qquad \epsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \tag{B1}$$

Equations (B1) imply the following compatibility condition on the strains

$$\frac{\partial^2 \gamma_{xy}}{\partial x \, \partial y} - \frac{\partial^2 \epsilon_x}{\partial y^2} - \frac{\partial^2 \epsilon_y}{\partial x^2} = 0 \tag{B2}$$

The plate equilibrium equations, namely

$$\frac{\partial N_x}{\partial x} + \frac{\partial N_{xy}}{\partial y} = 0 \qquad \frac{\partial N_y}{\partial y} + \frac{\partial N_{xy}}{\partial x} = 0 \tag{B3}$$

imply the existence of a stress function $F(x,y)$ such that

$$N_x = \partial^2 F/\partial y^2 \qquad N_y = \partial^2 F/\partial x^2 \qquad N_{xy} = - \partial^2 F/\partial x \partial y \tag{B4}$$

Eliminating the strains in equation (B2) by use of equations (2), and then the stresses by use of equations (B4) leads to the following form of the compatibility condition, in which account is already taken of the equilibrium and stress-strain relations:

$$C_2 \frac{\partial^4 F}{\partial x^4} + (C_4 - 2C_3) \frac{\partial^4 F}{\partial x^2 \partial y^2} + C_1 \frac{\partial^4 F}{\partial y^4} + \frac{\partial^2 e_y}{\partial x^2} + \frac{\partial^2 e_x}{\partial y^2} = 0 \tag{B5}$$

Considering now infinitesimal lengths of the stiffeners as free bodies, and utilizing the third of equations (B4) to express N_{xy} at the plate edges in terms of F, one obtains the following equilibrium equations governing the longitudinal variations of the stiffener cross-sectional tensions:

$$\begin{aligned} dP_1/dy - (\partial^2 F/\partial x \partial y)_{x=0} - q_1(y) &= 0 \\ dP_2/dy + (\partial^2 F/\partial x \partial y)_{x=a} + q_2(y) &= 0 \\ dP_3/dx - (\partial^2 F/\partial x \partial y)_{y=0} - q_3(x) &= 0 \\ dP_4/dx + (\partial^2 F/\partial x \partial y)_{y=b} + q_4(x) &= 0 \end{aligned} \tag{B6}$$

Integral attachment between the stiffeners and the plate edges implies equality of their longitudinal strains and leads to the following additional set of conditions on P_1, P_2, P_3, and P_4:

$$P_1(y)/(A_1E_1) + e_1(y) = \epsilon_y(0,y) = (e_y + C_2N_y - C_3N_x)_{x=0}$$

$$P_2(y)/(A_2E_2) + e_2(y) = \epsilon_y(a,y) = (e_y + C_2N_y - C_3N_x)_{x=a}$$

$$P_3(x)/(A_3E_3) + e_3(x) = \epsilon_x(x,0) = (e_x + C_1N_x - C_3N_y)_{y=0}$$

$$P_4(x)/(A_4E_4) + e_4(x) = \epsilon_x(x,b) = (e_x + C_1N_x - C_3N_y)_{y=b}$$

in which equations (2) have been used to obtain the right-hand terms. The assumption of negligible bending stiffness for the stiffeners permits the substitutions $(N_x)_{x=0} = N_1(y)$, etc. to be made in these terms. Expressing the remaining stress resultants in terms of F, one obtains

$$\begin{aligned} \frac{P_1(y)}{A_1\ E_1} + [e_1(y) - e_y(0,y)] - C_2\ \left(\frac{\partial^2 F}{\partial x^2}\right)_{x=0} + C_3N_1(y) &= 0 \\ \frac{P_2(y)}{A_2\ E_2} + [e_2(y) - e_y(a,y)] - C_2\ \left(\frac{\partial^2 F}{\partial x^2}\right)_{x=a} + C_3N_2(y) &= 0 \\ \frac{P_3(x)}{A_3\ E_3} + [e_3(x) - e_x(x,0)] - C_1\ \left(\frac{\partial^2 F}{\partial y^2}\right)_{y=0} + C_3N_3(x) &= 0 \\ \frac{P_4(x)}{A_4\ E_4} + [e_4(x) - e_x(x,b)] - C_1\ \left(\frac{\partial^2 F}{\partial y^2}\right)_{y=b} + C_3N_4(x) &= 0 \end{aligned} \tag{B7}$$

The problem can now be stated essentially as follows: to solve equations (B5), (B6), and (B7) for F, P_1, P_2, P_3, and P_4, subject to boundary conditions arising from the prescribed forces at the stiffener ends and the prescribed distributed loadings N_1, N_2, N_3, and N_4. In the following sections a formal solution to this problem will be obtained in terms of Fourier series.

Boundary values of F. - The fact that the distributed loadings N_1, N_2, N_3, and N_4 are transmitted directly to the plate means that the second derivative of F in the direction of the edge is known. Therefore two integrations will give the variation of F along each edge in terms of the unknown corner values and the known N_1, N_2, N_3, N_4. For example (using subscripts on F now for convenience to denote partial differentiation),

$$F_{yy}(0,y) = N_1(y)$$

Therefore

$$F_y(0,y) = F_y(0,0) + \int_0^y N_1(y')dy'$$

Therefore

$$F(0,y) = F(0,0) + yF_y(0,0) + \int_0^y \int_0^{y'} N_1(y'')dy''dy' \tag{B8}$$

Substitution of y = b in equation (B8) gives

$$F_y(0,0) = \frac{1}{b}\left[F(0,b) - F(0,0) - \int_0^b \int_0^{y'} N_1(y'')dy''dy'\right]$$

which result, substituted back into equation (B8), gives

$$F(0,y) = F(0,0) + \frac{y}{b}\left[F(0,b) - F(0,0) - \int_0^b \int_0^{y'} N_1(y'')dy''dy'\right] + \int_0^y \int_0^{y'} N_1(y'')dy''dy' \tag{B9}$$

Thus the variation of F along the edge x = 0 has been determined to within two constants, F(0,0) and F(0,b). Replacing $N_1(y'')$ by its series expansion, equation (3), and carrying out the integrations indicated in equation (B9) give

$$F(0,y) = F(0,0) + \frac{y}{b}\left[F(0,b) - F(0,0)\right] - \sum_{n=1}^{N} B'_n \left(\frac{b}{n\pi}\right)^2 \sin\frac{n\pi y}{b} \tag{B10}$$

going through a similar procedure for each of the remaining edges, one obtains

$$F(a,y) = F(a,0) + \frac{y}{b}\,[F(a,b) - F(a,0)] - \sum_{n=1}^{N} B''_n \left(\frac{b}{n\pi}\right)^2 \sin\frac{n\pi y}{b} \tag{B11}$$

$$F(x,0) = F(0,0) + \frac{x}{a}\,[F(a,0) - F(0,0)] - \sum_{m=1}^{M} B'''_m \left(\frac{a}{m\pi}\right)^2 \sin\frac{m\pi x}{a} \tag{B12}$$

$$F(x,b) = F(0,b) + \frac{x}{a}\,[F(a,b) - F(0,b)] - \sum_{m=1}^{M} B''''_m \left(\frac{a}{m\pi}\right)^2 \sin\frac{m\pi x}{a} \tag{B13}$$

For later use it will be necessary to have Fourier expansions of the boundary values of F in the form

$$\begin{aligned}
F(0,y) &= \sum_{n=1}^{N} a'_n \sin(n\pi y/b) \\
F(a,y) &= \sum_{n=1}^{N} a''_n \sin(n\pi y/b) \\
F(x,0) &= \sum_{m=1}^{M} a'''_m \sin(m\pi x/a) \\
F(x,b) &= \sum_{m=1}^{M} a''''_m \sin(m\pi x/a)
\end{aligned} \tag{B14}$$

Evaluating the coefficients in these series through the formulas $a'_n = (2/b)\int_0^b F(0,y)\sin(n\pi y/b)dy$, etc., with F(0,y), etc. replaced by the right-hand sides of equations (B10) to (B13), one obtains

$$\begin{aligned}
a'_n &= \frac{2}{n\pi}\,[F(0,0) - (-1)^n F(0,b)] - \left(\frac{b}{n\pi}\right)^2 B'_n \\
a''_n &= \frac{2}{n\pi}\,[F(a,0) - (-1)^n F(a,b)] - \left(\frac{b}{n\pi}\right)^2 B''_n \\
a'''_m &= \frac{2}{m\pi}\,[F(0,0) - (-1)^m F(a,0)] - \left(\frac{a}{m\pi}\right)^2 B'''_m \\
a''''_m &= \frac{2}{m\pi}\,[F(0,b) - (-1)^m F(a,b)] - \left(\frac{a}{m\pi}\right)^2 B''''_m
\end{aligned} \tag{B15}$$

Series assumptions for $F(x,y)$ and P_1, P_2, P_3, P_4. - In the interior of the plate (i.e. in the region $0 < x < a$, $0 < y < b$, excluding the edges $x = 0$, a and $y = 0$, b), the stress function $F(x,y)$ will be assumed to be representable by the double Fourier series

$$F(x,y) = \sum_{m=1}^{M} \sum_{n=1}^{N} a_{mn} \sin (m\pi x/a) \sin (n\pi y/b) \tag{B16}$$

with as yet unknown coefficients. Equation (B16) is, of course, not valid at the edges; however there the values of F are already expressed in series form by equations (B14) and (B15). Similarly the stiffener forces will be assumed in the form

$$\begin{aligned} P_1(y) &= \sum_{n=1}^{N} s'_n \sin (n\pi y/b) \\ P_2(y) &= \sum_{n=1}^{N} s''_n \sin (n\pi y/b) \\ P_3(x) &= \sum_{m=1}^{M} s'''_m \sin (m\pi x/a) \\ P_4(x) &= \sum_{m=1}^{M} s''''_m \sin (m\pi x/a) \end{aligned} \tag{B17}$$

for all cross sections except the end cross sections. At the end cross sections the stiffener forces are already known from the given loading (see fig. 1).

The coefficients in the series in equations (B16) and (B17) are related to the left-hand sides through the usual formulas

$$a_{mn} = \frac{4}{ab} \int_0^a \int_0^b F(x,y) \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \, dydx \tag{B18}$$

$$s'_n = \frac{2}{b} \int_0^b P_1(y) \sin \frac{n\pi y}{b} \, dy \quad , \quad \text{etc.} \tag{B19}$$

Series for the derivatives of $F(x,y)$ and P_1, P_2, P_3, P_4. - The derivatives appearing in equations (B4) to (B7) will also be assumed expressible in series as follows

$$dP_1/dy = \sum_{n=0}^{N} t'_n \cos(n\pi y/b) \qquad (0 \le y \le b)$$

$$dP_2/dy = \sum_{n=0}^{N} t''_n \cos(n\pi y/b) \qquad (0 \le y \le b)$$

$$dP_3/dx = \sum_{m=0}^{M} t'''_m \cos(m\pi x/a) \qquad (0 \le x \le a) \qquad \text{(B20)}$$

$$dP_4/dx = \sum_{m=0}^{M} t''''_m \cos(m\pi x/a) \qquad (0 \le x \le a)$$

$$N_y = \partial^2 F/\partial x^2 = \sum_{m=1}^{M} \sum_{n=1}^{N} c_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \quad \begin{pmatrix} 0 < x < a \\ 0 < y < b \end{pmatrix} \qquad \text{(B21)}$$

$$(N_y)_{x=0} = (\partial^2 F/\partial x^2)_{x=0} = \sum_{n=1}^{N} c'_n \sin(n\pi y/b) \qquad (0 < y < b) \qquad \text{(B22)}$$

$$(N_y)_{x=a} = (\partial^2 F/\partial x^2)_{x=a} = \sum_{n=1}^{N} c''_n \sin(n\pi y/b) \qquad (0 < y < b) \qquad \text{(B23)}$$

$$N_x = \partial^2 F/\partial y^2 = \sum_{m=1}^{M} \sum_{n=1}^{N} g_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \quad \begin{pmatrix} 0 < x < a \\ 0 < y < b \end{pmatrix} \qquad \text{(B24)}$$

$$(N_x)_{y=0} = (\partial^2 F/\partial y^2)_{y=0} = \sum_{m=1}^{M} g'_m \sin(m\pi x/a) \qquad (0 < x < a) \qquad \text{(B25)}$$

$$(N_x)_{y=b} = (\partial^2 F/\partial y^2)_{y=b} = \sum_{m=1}^{M} g''_m \sin(m\pi x/a) \qquad (0 < x < a) \qquad \text{(B26)}$$

$$-N_{xy} = \partial^2 F/\partial x \partial y = \sum_{m=0}^{M} \sum_{n=0}^{N} j_{mn} \cos(m\pi x/a) \cos(n\pi y/b) \quad \begin{pmatrix} 0 \le x \le a \\ 0 \le y \le b \end{pmatrix} \qquad \text{(B27)}$$

$$\partial^4 F/\partial x^4 = \sum_{m=1}^{M} \sum_{n=1}^{N} e_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \quad \begin{pmatrix} 0 < x < a \\ 0 < y < b \end{pmatrix} \qquad \text{(B28)}$$

$$\partial^4 F/\partial y^4 = \sum_{m=1}^{M} \sum_{n=1}^{N} i_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \quad \begin{pmatrix} 0 < x < a \\ 0 < y < b \end{pmatrix} \qquad \text{(B29)}$$

$$\partial^4 F/\partial x^2 \partial y^2 = \sum_{m=1}^{M} \sum_{n=1}^{N} p_{mn} \sin(m\pi x/a) \sin(n\pi y/b) \quad \begin{pmatrix} 0 < x < a \\ 0 < y < b \end{pmatrix} \qquad \text{(B30)}$$

where

$$t_n' = \frac{2 - \delta_{n0}}{b} \int_0^b (dP_1/dy) \cos (n\pi y/b)\, dy \text{ , etc.} \tag{B31}$$

$$c_{mn} = \frac{4}{ab} \int_0^a \int_0^b (\partial^2 F/\partial x^2) \sin (m\pi x/a) \sin (n\pi y/b)\, dydx \tag{B32}$$

$$c_n' = \frac{2}{b} \int_0^b (\partial^2 F/\partial x^2)_{x=0} \sin (n\pi y/b)\, dy \tag{B33}$$

$$c_n'' = \frac{2}{b} \int_0^b (\partial^2 F/\partial x^2)_{x=a} \sin (n\pi y/b)\, dy \tag{B34}$$

$$g_{mn} = \frac{4}{ab} \int_0^a \int_0^b (\partial^2 F/\partial y^2) \sin (m\pi x/a) \sin (n\pi y/b)\, dydx \tag{B35}$$

$$g_m' = \frac{2}{a} \int_0^a (\partial^2 F/\partial y^2)_{y=0} \sin (m\pi x/a)\, dx \tag{B36}$$

$$g_m'' = \frac{2}{a} \int_0^a (\partial^2 F/\partial y^2)_{y=b} \sin (m\pi x/a)\, dx \tag{B37}$$

$$j_{mn} = \frac{(2-\delta_{m0})(2-\delta_{n0})}{ab} \int_0^a \int_0^b (\partial^2 F/\partial x \partial y) \cos (m\pi x/a) \cos (n\pi y/b)\, dydx \tag{B38}$$

$$e_{mn} = \frac{4}{ab} \int_0^a \int_0^b (\partial^4 F/\partial x^4) \sin (m\pi x/a) \sin (n\pi y/b)\, dydx \tag{B39}$$

$$i_{mn} = \frac{4}{ab} \int_0^a \int_0^b (\partial^4 F/\partial y^4) \sin (m\pi x/a) \sin (n\pi y/b)\, dydx \tag{B40}$$

$$p_{mn} = \frac{4}{ab} \int_0^a \int_0^b (\partial^4 F/\partial x^2 \partial y^2) \sin (m\pi x/a) \sin (n\pi y/b)\, dydx \tag{B41}$$

The coefficients appearing in the series for the derivatives (equations (B20) to (B30)) are, of course, not independent of the coefficients in the series for the basic quantities (equations (B16) and (B17)). The former can be expressed in terms of the latter by means of integrations by parts in the right-hand sides of equations (B31) to (B41)*. For example, from equations (B31)

$$t'_n = \frac{2 - \delta_{n0}}{b} [P_1(b) \cos n\pi - P_1(0) + \frac{n\pi}{b} \int_0^b P_1(y) \sin \frac{n\pi y}{b} dy]$$

$$= \frac{2 - \delta_{n0}}{b} [P_1(b) \cos n\pi - P_1(0)] + \frac{n\pi}{b} s'_n$$

$$t''_n = \frac{2 - \delta_{n0}}{b} [P_2(b) \cos n\pi - P_2(0)] + \frac{n\pi}{b} s''_n \qquad \text{(B42)}$$

$$t'''_m = \frac{2 - \delta_{m0}}{a} [P_3(a) \cos m\pi - P_3(0)] + \frac{m\pi}{a} s'''_m$$

$$t''''_m = \frac{2 - \delta_{m0}}{a} [P_4(a) \cos m\pi - P_4(0)] + \frac{m\pi}{a} s''''_m$$

Similarly two partial integrations with respect to x in equation (B32) give

$$c_{mn} = \frac{m\pi}{a} \frac{2}{a} [a'_n - (-1)^m a''_n] - (\frac{m\pi}{a})^2 a_{mn} \qquad \text{(B43)}$$

Two with respect to y in equation (B35) give

$$g_{mn} = \frac{n\pi}{b} \frac{2}{b} [a'''_m - (-1)^n a''''_m] - (\frac{n\pi}{b})^2 a_{mn} \qquad \text{(B44)}$$

In equation (B38) partial integration with respect to x, followed by partial integration with respect to y in both of the resulting terms, gives

$$j_{mn} = \frac{(2-\delta_{m0})(2-\delta_{n0})}{ab} [(-1)^{m+n} F(a,b)-(-1)^m F(a,0)-(-1)^n F(0,b) + F(0,0)]$$

$$+ \frac{m\pi}{a} \frac{2-\delta_{n0}}{b} [(-1)^n a''''_m - a'''_m] + \frac{n\pi}{b} \frac{2-\delta_{m0}}{a} [(-1)^m a''_n - a'_n] + \frac{m\pi}{a} \frac{n\pi}{b} a_{mn} \qquad \text{(B45)}$$

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*Such a technique was employed for plate bending problems by A. E. Green (reference 1), who ascribes its earlier use to S. Goldstein.

in which single-valuedness of F at the corners has been assumed. Proceeding in a similar fashion with the right-hand sides of equations (B39) to (B41), one obtains

$$e_{mn} = \frac{m\pi}{a}\frac{2}{a}[c_n' - (-1)^m c_n''] - \left(\frac{m\pi}{a}\right)^3 \frac{2}{a}[a_n' - (-1)^m a_n''] + \left(\frac{m\pi}{a}\right)^4 a_{mn} \tag{B46}$$

$$i_{mn} = \frac{n\pi}{b}\frac{2}{b}[g_m' - (-1)^n g_m''] - \left(\frac{n\pi}{b}\right)^3 \frac{2}{b}[a_m''' - (-1)^n a_m''''] + \left(\frac{n\pi}{b}\right)^4 a_{mn} \tag{B47}$$

$$p_{mn} = \frac{4}{ab}\frac{m\pi}{a}\frac{n\pi}{b}[(-1)^{m+n} F(a,b) - (-1)^m F(a,0) - (-1)^n F(0,b) + F(0,0)]$$

$$+ \frac{2}{b}\left(\frac{m\pi}{a}\right)^2 \frac{n\pi}{b}[(-1)^n a_m'''' - a_m'''] + \frac{2}{a}\left(\frac{n\pi}{b}\right)^2 \frac{m\pi}{a}[(-1)^m a_n'' - a_n'] + \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 a_{mn} \tag{B48}$$

Substitution of series into the basic equations. - Through equations (B42) to (B48) all the unknown coefficients in the derivative series are expressed in terms of the basic unknowns a_{mn}; c_n', c_n'', g_m', g_m''; s_n', s_n'', s_m''', s_m''''; F(a,b), F(a,0), F(0,b), and F(0,0). Relationships among these unknowns will now be obtained by substituting the assumed series into the basic equations (B5), (B6), and (B7).

Considering first equation (B5), substituting into it the series expansions from equations (6) and (B28) to (B30), and eliminating e_{mn}, i_{mn}, and p_{mn} through equations (B46) to (B48), one obtains

$$C_2\left\{\frac{m\pi}{a}\frac{2}{a}[c_n' - (-1)^m c_n''] - \left(\frac{m\pi}{a}\right)^3 \frac{2}{a}[a_n' - (-1)^m a_n''] + \left(\frac{m\pi}{a}\right)^4 a_{mn}\right\}$$

$$+ (C_4 - 2C_3)\left\{\frac{4}{ab}\frac{m\pi}{a}\frac{n\pi}{b}[(-1)^{m+n}F(a,b)-(-1)^m F(a,0)-(-1)^n F(0,b) + F(0,0)]\right.$$

$$\left.+ \frac{2}{b}\left(\frac{m\pi}{a}\right)^2 \frac{n\pi}{b}[(-1)^n a_m'''' - a_m'''] + \frac{2}{a}\left(\frac{n\pi}{b}\right)^2 \frac{m\pi}{a}[(-1)^m a_n''-a_n'] + \left(\frac{m\pi}{a}\right)^2\left(\frac{n\pi}{b}\right)^2 a_{mn}\right\}$$

$$+ C_1\left\{\frac{n\pi}{b}\frac{2}{b}[g_m' - (-1)^n g_m''] - \left(\frac{n\pi}{b}\right)^3 \frac{2}{b}[a_m''' - (-1)^n a_m''''] + \left(\frac{n\pi}{b}\right)^4 a_{mn}\right\} + T_{mn} = 0$$

Solving this equation for a_{mn} and eliminating a_n', a_n'', a_m''', and a_m'''' through equations (B15), one obtains

$$a_{mn} = \frac{4}{mn\,\pi^2}\,[(-1)^{m+n}\,F(a,b) - (-1)^m\,F(a,0) - (-1)^n\,F(0,b) + F(0,0)]$$

$$- \frac{1}{E_{mn}}\Big\{T_{mn} + \frac{2}{a}\frac{m\pi}{a}\,[c_n' - (-1)^m\,c_n'']\,C_2 + \frac{2}{b}\frac{n\pi}{b}\,[g_m' - (-1)^n\,g_m'']\,C_1$$

$$+ \frac{2}{a}\frac{m\pi}{a}\,(\frac{b}{n\pi})^2[B_n' - (-1)^m\,B_n''][(\frac{m\pi}{a})^2\,C_2 + (\frac{n\pi}{b})^2\,(C_4 - 2C_3)]$$

$$+ \frac{2}{b}\frac{n\pi}{b}\,(\frac{a}{m\pi})^2\,[B_m''' - (-1)^n\,B_m''''][(\frac{n\pi}{b})^2\,C_1 + (\frac{m\pi}{a})^2(C_4 - 2C_3)]\Big\} \tag{B49}$$

where

$$E_{mn} = C_2\,(\frac{m\pi}{a})^4 + (C_4 - 2C_3)\,(\frac{m\pi}{a})^2(\frac{n\pi}{b})^2 + C_1\,(\frac{n\pi}{b})^4 \tag{B50}$$

Thus through the compatibility equation the unknown a_{mn} have been expressed in terms of a smaller class of unknowns, namely the c_n', c_n'', g_m' and g_m''.

Turning now to the stiffener equilibrium equations (B6), substituting the series from equations (B20), (B27), and (4), and utilizing equations (B42), one obtains the relationships

$$\frac{2 - \delta_{n0}}{b}\,[(-1)^n\,P_1(b) - P_1(0)] + \frac{n\pi}{b}\,s_n' - Q_n' - \sum_{m=0}^{M} j_{mn} = 0 \quad (n=0,1,\ldots,N) \tag{B51}$$

$$\frac{2 - \delta_{n0}}{b}\,[(-1)^n\,P_2(b) - P_2(0)] + \frac{n\pi}{b}\,s_n'' + Q_n'' + \sum_{m=0}^{M} j_{mn}(-1)^m = 0 \quad (n=0,1,\ldots,N) \tag{B52}$$

$$\frac{2 - \delta_{m0}}{a}\,[(-1)^m\,P_3(a) - P_3(0)] + \frac{m\pi}{a}\,s_m''' - Q_m''' - \sum_{n=0}^{N} j_{mn} = 0 \quad (m=0,1,\ldots,M) \tag{B53}$$

$$\frac{2 - \delta_{m0}}{a}\,[(-1)^m\,P_4(a) - P_4(0)] + \frac{m\pi}{a}\,s_m'''' + Q'''' + \sum_{n=0}^{N} j_{mn}(-1)^n = 0 \quad (m=0,1,\ldots M) \tag{B54}$$

The n = 0 and m = 0 equations of this group will be studied first. From equation (B45), in conjunction with (B15) and (B49), it is noted that

$$j_{00} = \frac{1}{ab} [F(0,0) - F(a,0) - F(0,b) + F(a,b)] \tag{B55}$$

$$j_{0n} = \frac{1}{a} \frac{b}{n\pi} (B'_n - B''_n) \quad \text{for} \quad n \neq 0$$

$$j_{m0} = \frac{1}{b} \frac{a}{m\pi} (B'''_m - B''''_m) \quad \text{for} \quad m \neq 0 \tag{B56}$$

Therefore equations (B51) and (B52) for n = 0, (B53) and (B54) for m = 0 give

$$P_1(b) - P_1(0) - bQ'_0 - bj_{00} + \sum_{m=1}^{M} (a/m\pi)(B''''_m - B'''_m) = 0 \tag{B57a}$$

$$P_2(b) - P_2(0) + bQ''_0 + bj_{00} - \sum_{m=1}^{M} (a/m\pi)(B''''_m - B'''_m)(-1)^m = 0 \tag{B57b}$$

$$P_3(a) - P_3(0) - aQ'''_0 - aj_{00} + \sum_{n=1}^{N} (b/n\pi)(B''_n - B'_n) = 0 \tag{B57c}$$

$$P_4(a) - P_4(0) + aQ''''_0 + aj_{00} - \sum_{n=1}^{N} (b/n\pi)(B''_n - B'_n)(-1)^n = 0 \tag{B57d}$$

Three of equations (B57) are redundant if the applied loading constitutes an equilibrium system for the structure as a whole: It is easily verified that the equation of equilibrium of forces in the x-direction is the same as that obtained by adding equations (c) and (d), the equation of equilibrium for the y-direction the same as the sum of equations (a) and (b), and the equation of equilibrium of moments about the center of the plate equivalent to (a/2)[eq.(a) - eq.(b)] + (b/2)[eq.(d)-eq.(c)]. Hence any three of equations (B57) may be eliminated from further consideration. The one remaining equation serves to establish the value of j_{00}. Selecting equation (a) as this remaining one gives

$$j_{00} = -Q'_0 + \frac{1}{b}\left\{P_1(b) - P_1(0) + \sum_{m=1}^{M} \frac{a}{m\pi} (B''''_m - B'''_m)\right\} \tag{B58}$$

Three alternate expressions for j_{00} can be obtained from equations (b), (c), and (d). For the sake of symmetry one can sum all four expressions for j_{00} and divide by four in order to obtain still another formula for j_{00}; while esthetically more satisfying, this formula is more complicated than equation (B58).

Finally, equations (B7), with the various terms replaced by their series expansions from equations (3), (5), (B22), (B23), (B25), and (B26), give

$$s_n'/(A_1E_1) + T_n' - C_2c_n' + C_3B_n' = 0$$

$$s_n''/(A_2E_2) + T_n'' - C_2c_n'' + C_3B_n'' = 0$$

$$s_m'''/(A_3E_3) + T_m''' - C_1g_m' + C_3B_m''' = 0 \tag{B59}$$

$$s_m''''/(A_4E_4) + T_m'''' - C_1g_m'' + C_3B_m'''' = 0$$

In obtaining equations (B59), the assumption of uniform stiffeners was used for the first time.

Reduction in the number of simultaneous equations. - Equations (B51) to (B54), with the n = 0 and m = 0 equations excluded, can be used to obtain a system of simultaneous equations in which the c_n', c_n'', g_m' and g_m'' are the only unknowns. This is accomplished by eliminating s_n', s_n'', s_m''', and s_m'''' with the aid of equations (B59), and j_{mn} by means of the following expression

$$\begin{aligned} j_{mn} = &- \frac{mn\pi^2}{ab\ E_{mn}} \left\{T_{mn} + \frac{2}{a}\frac{m\pi}{a}\left[c_n' - (-1)^m c_n''\right] C_2 + \frac{2}{b}\frac{n\pi}{b}\left[g_m' - (-1)^n g_m''\right]C_1\right\} \\ \begin{pmatrix} m\neq 0 \\ n\neq 0\end{pmatrix} & \\ &+ \frac{1}{E_{mn}}\left\{\frac{2}{b}\left(\frac{m\pi}{a}\right)^3 \left[B_m''' - (-1)^n B_m''''\right] C_2 + \frac{2}{a}\left(\frac{n\pi}{b}\right)^3\left[B_n' - (-1)^m B_n''\right]C_1\right\} \end{aligned} \tag{B60}$$

which is obtained from equations (B45), (B15), and (B49). The resulting system of simultaneous equations is

$$\begin{aligned} &\bar{c}_n'\left[A_1E_1 + \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}}\right] - \bar{c}_n''\left[\frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2(-1)^m}{E_{mn}}\right] \\ &\qquad = R_n' - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\sum_{m=1}^{M}\frac{\bar{g}_m' - (-1)^n\bar{g}_m''}{E_{mn}} \quad (n=1,2,\ldots,N) \end{aligned} \tag{B61}$$

$$\begin{aligned} &-\bar{c}_n'\left[\frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2(-1)^m}{E_{mn}}\right] + \bar{c}_n''\left[A_2E_2 + \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}}\right] \\ &\qquad = R_n'' + \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\sum_{m=1}^{M}(-1)^m\frac{\bar{g}_m' - (-1)^n\bar{g}_m''}{E_{mn}} \quad (n=1,2,\ldots,N) \end{aligned} \tag{B62}$$

$$\bar{g}_m' \left[A_3E_3 + \frac{2}{b}\sum_{n=1}^{N}\frac{(n\pi/b)^2}{E_{mn}}\right] - \bar{g}_m''\left[\frac{2}{b}\sum_{n=1}^{N}\frac{(n\pi/b)^2\,(-1)^n}{E_{mn}}\right]$$

$$= R_m''' - \frac{2}{a}\left(\frac{m\pi}{a}\right)^2\sum_{n=1}^{N}\frac{\bar{c}_n' - (-1)^m\,\bar{c}_n''}{E_{mn}} \quad (m=1,2,\ldots,M) \qquad \text{(B63)}$$

$$-\bar{g}_m'\left[\frac{2}{b}\sum_{n=1}^{N}\frac{(n\pi/b)^2\,(-1)^n}{E_{mn}}\right] + \bar{g}_m''\left[A_4E_4 + \frac{2}{b}\sum_{n=1}^{N}\frac{(n\pi/b)^2}{E_{mn}}\right]$$

$$= R_m'''' + \frac{2}{a}\left(\frac{m\pi}{a}\right)^2\sum_{n=1}^{N}(-1)^n\,\frac{\bar{c}_n - (-1)^m\,\bar{c}_n''}{E_{mn}} \quad (m=1,2,\ldots,M)$$

(B64)

where

$$\bar{c}_n' = c_n'C_2\,n\pi/b \quad , \quad \bar{c}_n'' = c_n''C_2\,n\pi/b \;,\; \bar{g}_m' = g_m'C_1\,m\pi/a \;,\; \bar{g}_m'' = g_m''C_1\,m\pi/a \qquad \text{(B65)}$$

and R_n', R_n'', R_m''', R_m'''' are the following combinations of known thermal and loading quantities:

$$R_n' = Q_n' + \frac{2}{b}[P_1(0)-(-1)^n\,P_1(b)] + \frac{b}{an\pi}(B_n'-B_n'') + A_1E_1\,\frac{n\pi}{b}(C_3B_n'+T_n') - \sum_{m=1}^{M}K_{mn}$$

$$R_n'' = -Q_n'' + \frac{2}{b}[P_2(0)-(-1)^n\,P_2(b)] - \frac{b}{an\pi}(B_n'-B_n'') + A_2E_2\,\frac{n\pi}{b}(C_3B_n''+T_n'') + \sum_{m=1}^{M}(-1)^mK_{mn}$$

(B66)

$$R_m''' = Q_m''' + \frac{2}{a}[P_3(0)-(-1)^m\,P_3(a)] + \frac{a}{bm\pi}(B_m'''-B_m'''') + A_3E_3\,\frac{m\pi}{a}(C_3B_m''' + T_m''') - \sum_{n=1}^{N}K_{mn}$$

$$R_m'''' = -Q_m'''' + \frac{2}{a}[P_4(0)-(-1)^mP_4(a)] - \frac{a}{bm\pi}(B_m'''-B_m'''') + A_4E_4\,\frac{m\pi}{a}(C_3B_m''''+T_m'''') + \sum_{n=1}^{N}(-1)^nK_{mn}$$

and

$$K_{mn} = \frac{1}{E_{mn}}\left\{\frac{mn\pi^2}{ab}\,T_{mn} - \frac{2}{b}\left(\frac{m\pi}{a}\right)^3C_2\,[B_m'''-(-1)^n\,B_m''''] - \frac{2}{a}\left(\frac{n\pi}{b}\right)^3C_1[B_n'-(-1)^mB_n'']\right\} \qquad \text{(B67)}$$

Equations (B61) to (B65) can be solved simultaneously for the c_n', c_n'', g_m', and g_m''. With these known, equations (B59) will furnish the values of s_n', s_n'', s_m''', s_m'''', and equations (B60), (B58), and (B56) the values of the

j_{mn}. Equations (B17) will then give the stiffener stresses, equations (B21) to (B27) the plate stresses.

Further reduction in the number of simultaneous equations. - Equations (B61) and (B62), written for the same value of n, can be solved for each $\bar{c}'_n$ and $\bar{c}''_n$ in terms of all the $\bar{g}'_m$ and $\bar{g}''_m$, with the result

$$\bar{c}'_n = \frac{1}{\Delta_n}\left\{\left[A_2E_2 + \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}}\right]\left[R'_n - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\sum_{m=1}^{M}\frac{\bar{g}'_m - (-1)^n\,\bar{g}''_m}{E_{mn}}\right]\right.$$
$$\left. + \left[\frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2(-1)^m}{E_{mn}}\right]\left[R''_n + \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\sum_{m=1}^{M}(-1)^m\,\frac{\bar{g}'_m - (-1)^n\,\bar{g}''_m}{E_{mn}}\right]\right\} \tag{B68}$$

$$\bar{c}''_n = \frac{1}{\Delta_n}\left\{\left[A_1E_1 + \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}}\right]\left[R''_n + \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\sum_{m=1}^{M}(-1)^m\,\frac{\bar{g}'_m-(-1)^n\,\bar{g}''_m}{E_{mn}}\right]\right.$$
$$\left. + \left[\frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2(-1)^m}{E_{mn}}\right]\left[R'_n - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\sum_{m=1}^{M}\frac{\bar{g}'_m - (-1)^n\,\bar{g}''_m}{E_{mn}}\right]\right\}$$

where

$$\Delta_n = \left[A_1E_1 + \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}}\right]\left[A_2E_2 + \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}}\right] - \left[\frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2(-1)^m}{E_{mn}}\right]^2$$

Using equations (B68) to eliminate the $\bar{c}'_n$ and $\bar{c}''_n$ in equations (B63) and (B64), one obtains the following system of simultaneous equations involving only the $\bar{g}'_m$ and $\bar{g}''_m$ as unknowns:

$$\sum_{p=1}^{M}\left[\bar{g}'_p\left(W'_{MNmp} - \delta_{mp}X'''_{Nm}\right) + \bar{g}''_p\left(W''_{MNmp} + \delta_{mp}Y''_{Nm}\right)\right] = -U'_{mMN}$$
$$\sum_{p=1}^{M}\left[\bar{g}'_p\left(W''_{MNmp} + \delta_{mp}Y''_{Nm}\right) + \bar{g}''_p\left(W'_{MNmp} - \delta_{mp}X''''_{Nm}\right)\right] = -U''_{mMN} \tag{B70}$$

$$(m = 1, 2, \ldots, M)$$

where

$$U'_{mMN} = R'''_m - \frac{2}{a} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{L_{Mn}}{E_{mn} \Delta_n}$$

$$U''_{mMN} = R''''_m + \frac{2}{a} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{(-1)^n L_{Mn}}{E_{mn} \Delta_n} \tag{B71}$$

$$L_{Mn} = X''_{Mn} R'_n + Y'_{Mn} R''_n - (-1)^m \left[X'_{Mn} R''_n + Y'_{Mn} R'_n \right] \tag{B72}$$

$$X'_{Mn} = A_1 E_1 + \frac{2}{a} \sum_{p=1}^{M} \frac{(p\pi/a)^2}{E_{pn}}$$

$$X''_{Mn} = A_2 E_2 + \frac{2}{a} \sum_{p=1}^{M} \frac{(p\pi/a)^2}{E_{pn}}$$

$$X'''_{Nm} = A_3 E_3 + \frac{2}{b} \sum_{n=1}^{N} \frac{(n\pi/b)^2}{E_{mn}} \tag{B73}$$

$$X''''_{Nm} = A_4 E_4 + \frac{2}{b} \sum_{n=1}^{N} \frac{(n\pi/b)^2}{E_{mn}}$$

$$Y'_{Mn} = \frac{2}{a} \sum_{p=1}^{M} \frac{(p\pi/a)^2 (-1)^p}{E_{pn}}$$

$$Y''_{Nm} = \frac{2}{b} \sum_{n=1}^{N} \frac{(n\pi/b)^2 (-1)^n}{E_{mn}} \tag{B74}$$

$$W'_{MNmp} = \frac{4}{ab} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{(n\pi/b)^2}{E_{mn} E_{pn} \Delta_n} V_{Mmnp}$$

$$W''_{MNmp} = -\frac{4}{ab} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{(-1)^n (n\pi/b)^2}{E_{mn} E_{pn} \Delta_n} V_{Mmnp} \tag{B75}$$

$$V_{Mmnp} = A_2E_2 + A_1E_1(-1)^{m+p} + \frac{8}{a} \sum_{q=1}^{M} \frac{1}{E_{qn}} \left(\frac{q\pi}{a}\right)^2 \delta_{mpq} \tag{B76}$$

$$\delta_{mpq} = \frac{1}{4}\left[1 - (-1)^{p+q} - (-1)^{q+m} + (-1)^{m+p}\right] = \begin{cases} 1 & \text{for m and p both even, q odd} \\ 1 & \text{for m and p both odd, q even} \\ 0 & \text{otherwise} \end{cases} \tag{B77}$$

The advantage of this preceding step is evident: Whereas the original system, equations (B61) to (B64), requires the solution of 2N + 2M simultaneous equations, the reduced system, equations (B70), contains only 2M simultaneous equations. Thus N may be taken arbitrarily large without increasing the number of simultaneous equations that have to be solved.

Special case: symmetry about $y = b/2$. - If the structure, loading, and thermal strains are symmetrical about the line $y = b/2$, then a corresponding symmetry obtains in the stress function F and in the plate and stiffener stresses. Consequently one may set

$$\begin{aligned} \bar{c}'_n &= \bar{c}''_n = 0 \qquad \text{for} \qquad n \text{ even} \\ \bar{g}'_m &= \bar{g}''_m \qquad \text{for} \qquad \text{all } m \end{aligned} \tag{B78}$$

In place of equations (B61) to (B64) the following system results:

$$\bar{c}'_n \left[A_1E_1 + \frac{2}{a} \sum_{m=1}^{M} \frac{(m\pi/a)^2}{E_{mn}}\right] - \bar{c}''_n \left[\frac{2}{a} \sum_{m=1}^{M} \frac{(m\pi/a)^2(-1)^m}{E_{mn}}\right]$$

$$= R'_n - \frac{4}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M} (\bar{g}'_m/E_{mn}) \tag{B61'}$$

$$(n = 1, 3, \ldots, N)$$

$$-\bar{c}'_n \left[\frac{2}{a} \sum_{m=1}^{M} \frac{(m\pi/a)^2(-1)^m}{E_{mn}}\right] + \bar{c}''_n \left[A_2E_2 + \frac{2}{a} \sum_{m=1}^{M} \frac{(m\pi/a)^2}{E_{mn}}\right]$$

$$= \bar{R}''_n + \frac{4}{b}\left(\frac{n\pi}{b}\right)^2 \sum_{m=1}^{M} (-1)^m (\bar{g}'_m/E_{mn}) \tag{B62'}$$

$$(n = 1, 3, \ldots, N)$$

$$\bar{g}'_m \left[A_3 E_3 + \frac{4}{b} \sum_{n=1,3,..}^{N} \frac{(n\pi/b)^2}{E_{mn}}\right]$$

$$= R'''_m - \frac{2}{a} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1,3,..}^{N} \frac{\bar{c}'_n - (-1)^m \bar{c}''_n}{E_{mn}} \qquad \text{(B63')}$$

$$(m = 1,2,\ldots, M)$$

where N is now restricted to odd integers.

Correspondingly, in place of equations (B70) the following system is obtained.

$$\sum_{p=1}^{M} \bar{g}'_p \left(2W'_{MNmp} + \delta_{mp} Y''_{Nm} - \delta_{mp} X'''_{Nm}\right) = - U'_{mMN} \qquad \text{(B70')}$$

$$(m = 1,2,\ldots, M)$$

where the primed symbols are defined as before through equations (B71), (B74) and (B75), but with the understanding that wherever a summation with respect to n appears in these definitions n shall now take on only odd values. The number of equations in the system (B70') is M, regardless of the value selected for N.

<u>Special case: symmetry about y = b/2 and x = a/2.</u> - A further reduction in the number of simultaneous equation results if the structure, loading, and thermal strains are symmetrical about both centerlines, y = b/2 and x = a/2. In this case

$$\begin{aligned} \bar{c}'_n &= \bar{c}''_n = 0 && \text{for} \quad n \quad \text{even} \\ \bar{g}'_m &= \bar{g}''_m = 0 && \text{for} \quad m \quad \text{even} \\ \bar{c}'_n &= \bar{c}''_n && \text{for} \quad n \quad \text{odd} \\ \bar{g}'_m &= \bar{g}''_m && \text{for} \quad m \quad \text{odd} \end{aligned} \qquad \text{(B79)}$$

Equations (B61') to (B63') are replaced by

$$\bar{c}'_n \left[A_1E_1 + \frac{4}{a} \sum_{m=1,3,..}^{M} \frac{(m\pi/a)^2}{E_{mn}}\right] = R'_n - \frac{4}{b} \left(\frac{n\pi}{b}\right)^2 \sum_{m=1,3,..}^{M} (\bar{g}'_m/E_{mn}) \qquad \text{(B61")}$$

$$(n=1,3,\ldots, N)$$

$$\bar{g}'_m \left[A_3E_3 + \frac{4}{b} \sum_{n=1,3,..}^{N} \frac{(n\pi/b)^2}{E_{mn}}\right] = R'''_m - \frac{4}{a} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1,3,..}^{N} (\bar{c}'_n/E_{mn}) \qquad \text{(B63")}$$

$$(m=1,3,\ldots, M)$$

where M and N are now restricted to odd integers.

Eliminating the $\bar{c}'_n$ in equation (B63") by means of equations (B61") gives the following system of simultaneous equations in the $\bar{g}'_m$ alone, which takes the place of equations (B70'):

$$\sum_{p=1,3,..}^{M} \bar{g}'_p \left\{\frac{16}{ab} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1,3,..}^{N} \left(\frac{n\pi}{b}\right)^2 (E_{mn}E_{pn}Z'_{Mn})^{-1} - \delta_{mp} Z'''_{Nm}\right\}$$

$$= - \left\{R'''_m - \frac{4}{a} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1,3,..}^{N} R'_n \, (E_{mn}Z'_{Mn})^{-1}\right\} \qquad \text{(B70")}$$

$$(m=1,3,\ldots, M)$$

where

$$\begin{aligned} Z'_{Mn} &= A_1E_1 + \frac{4}{a} \sum_{m=1,3,..}^{M} \frac{(m\pi/a)^2}{E_{mn}} \\ Z'''_{Nm} &= A_3E_3 + \frac{4}{b} \sum_{n=1,3,..}^{N} \frac{(n\pi/b)^2}{E_{mn}} \end{aligned} \qquad \text{(B73")}$$

Regardless of how large a value is assigned to N, there are only (M+1)/2 simultaneous equations in the system (B70").

Limiting case of large stiffener areas. - The case in which the stiffener cross-sectional areas are large compared to the plate cross-sectional area is of practical and theoretical interest. In order to study this case, let it be assumed that A_1E_1, A_2E_2, A_3E_3 and A_4E_4 all approach infinity while maintaining constant ratios with each other. Then equations (B61) to (B64) degenerate to

$$\bar{c}'_n = R'_n/(A_1E_1) \qquad \bar{c}''_n = R''_n/(A_2E_2)$$

$$\bar{g}'_m = R'''_m/(A_3E_3) \qquad \bar{g}''_m = (R''''_m/(A_4E_4) \tag{B80}$$

Thus for this limiting case it is unnecessary to solve simultaneous equations.

Equations (B80) represent a first-order perturbation solution of equations (B61) to (B64), in which the coefficients of $R'_n/(A_1E_1)$, $R''_n/(A_2E_2)$, etc. are expanded in series of powers of $(a^3\ E_{11}A_1E_1)^{-1}$ and only the zeroth power retained. If terms of the zeroth and first degree are retained, the following more accurate results are arrived at:

$$\bar{c}'_n = \frac{R'_n}{A_1E_1}\left[1 - \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}\ A_1E_1}\right] + \frac{R''_n}{A_2E_2}\ \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2(-1)^m}{E_{mn}\ A_1E_1}$$

$$- \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\sum_{m=1}^{M}\frac{1}{E_{mn}\ A_1E_1}\left(\frac{R'''_m}{A_3E_3}\right) + \frac{2}{b}\left(\frac{n\pi}{b}\right)^2(-1)^n\sum_{m=1}^{M}\frac{1}{E_{mn}\ A_1E_1}\left(\frac{R''''_m}{A_4E_4}\right) \tag{B81}$$

$$\bar{c}''_n = \frac{R'_n}{A_1E_1}\ \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2(-1)^m}{E_{mn}\ A_2E_2} + \frac{R''_n}{A_2E_2}\left[1 - \frac{2}{a}\sum_{m=1}^{M}\frac{(m\pi/a)^2}{E_{mn}\ A_2E_2}\right]$$

$$+ \frac{2}{b}\left(\frac{n\pi}{b}\right)^2\sum_{m=1}^{M}\frac{(-1)^m}{E_{mn}\ A_2E_2}\left(\frac{R'''_m}{A_3E_3}\right) - \frac{2}{b}\left(\frac{n\pi}{b}\right)^2(-1)^n\sum_{m=1}^{M}\frac{(-1)^m}{E_{mn}\ A_2E_2}\left(\frac{R''''_m}{A_4E_4}\right) \tag{B82}$$

$$\bar{g}'_m = \frac{R'''_m}{A_3E_3}\left[1 - \frac{2}{b}\sum_{n=1}^{N}\frac{(n\pi/b)^2}{E_{mn}\ A_3E_3}\right] + \frac{R''''_m}{A_4E_4}\ \frac{2}{b}\sum_{n=1}^{N}\frac{(n\pi/b)^2(-1)^n}{E_{mn}\ A_3E_3}$$

$$- \frac{2}{a}\left(\frac{m\pi}{a}\right)^2\sum_{n=1}^{N}\frac{1}{E_{mn}\ A_3E_3}\left(\frac{R'_n}{A_1E_1}\right) + \frac{2}{a}\left(\frac{m\pi}{a}\right)^2(-1)^m\sum_{n=1}^{N}\frac{1}{E_{mn}\ A_3E_3}\left(\frac{R''_n}{A_2E_2}\right) \tag{B83}$$

$$\bar{g}''_m = \frac{R'''_m}{A_3E_3} \frac{2}{b} \sum_{n=1}^{N} \frac{(n\pi/b)^2(-1)^n}{E_{mn} A_4E_4} + \frac{R''''_m}{A_4E_4} \left[1 - \frac{2}{b} \sum_{n=1}^{N} \frac{(n\pi/b)^2}{E_{mn} A_4E_4} \right]$$

$$+ \frac{2}{a} \left(\frac{m\pi}{a}\right)^2 \sum_{n=1}^{N} \frac{(-1)^n}{E_{mn} A_4E_4} \left(\frac{R'_n}{A_1E_1}\right) - \frac{2}{a} \left(\frac{m\pi}{a}\right)^2 (-1)^m \sum_{n=1}^{N} \frac{(-1)^n}{E_{mn} A_4E_4} \left(\frac{R''_n}{A_2E_2}\right) \tag{B84}$$

It is seen that a solution to this degree of approximation still retains the simplicity of equations (B80), in the sense that it is unnecessary to solve simultaneous equations to determine the $\bar{c}'_n$, $\bar{c}''_n$, $\bar{g}'_m$ and $\bar{g}''_m$. The coefficients of $R'_n/(A_1E_1)$, etc. in equations (B81) to (B84) are correct to terms of the first degree in $(a^3E_{11}A_1E_1)^{-1}$. To obtain expressions of greater accuracy it would once again be necessary to solve simultaneous equations.

APPENDIX C

ANALYSIS FOR THE CASE OF CONSTANT-STRESS (OR PRESCRIBED-STRESS) STIFFENERS

If the stiffener cross sections are varied so as to produce in each stiffener a constant (or non-constant but prescribed) stress everywhere along its length, then the determination of the plate stresses becomes extremely simple, for in that case the normal stress in the adjacent plate in a direction parallel to the stiffener is essentially given, and consequently the c_n', c_n'', g_m', g_m'' in equations (B22), (B23), (B25), and (B26) are known. (This is in contrast to the constant-area case, in which the c_n', etc. were unknown.) With these quantities known, equations (B60), (B58), (B56), and then (B21), (B24), and (B27) will furnish the remaining plate stresses.

To carry out the foregoing analysis in detail, let $\sigma_1(y)$, $\sigma_2(y)$, $\sigma_3(x)$, $\sigma_4(x)$ denote the prescribed, perhaps constant, values of the stresses (P_i/A_i) in the stiffeners at x = 0, x = a, y = 0, and y = b respectively. Then the strains in these stiffeners are known from equations (1). Equating these to the corresponding plate strains adjacent to the stiffener, equations (2), gives

$$\left.\begin{aligned}
e_1 + (\sigma_1/E_1) &= e_y(0,y) + C_2N_y(0,y) - C_3N_1(y)\\
e_2 + (\sigma_2/E_2) &= e_y(a,y) + C_2N_y(a,y) - C_3N_2(y)\\
e_3 + (\sigma_3/E_3) &= e_x(x,0) + C_1N_x(x,0) - C_3N_3(x)\\
e_4 + (\sigma_4/E_4) &= e_x(x,b) + C_1N_x(x,b) - C_3N_4(x)
\end{aligned}\right\} \qquad \text{(C1)}$$

whence

$$N_y(0,y) = \left(\frac{\partial^2F}{\partial x^2}\right)_{x=0} = \frac{1}{C_2}\left[e_1(y) - e_y(0,y) + \frac{\sigma_1(y)}{E_1} + C_3N_1(y)\right] \qquad \text{(C2)}$$

etc.

Substituting the known right-hand sides of equations (C2) into equations (B33), (B34), (B36), and (B37), and carrying out the integrations, one obtains the values of the c_n', c_n'', g_m', and g_m'', from which the plate stresses can be found as indicated in the previous paragraph.

In order to determine the variations of stiffener cross-sectional area required to produce the prescribed stiffener stresses, the relationships

$$A_1(y) = P_1(y)/\sigma_1(y) \qquad A_2(y) = P_2(y)/\sigma_2(y)$$
$$A_3(x) = P_3(x)/\sigma_3(x) \qquad A_4(x) = P_4(x)/\sigma_4(x) \tag{C3}$$

may be used, with $P_1(y)$ etc. evaluated by integrating the running loads on the stiffeners, starting at one end, as shown below.

$$P_1(y) = P_1(0) + \int_0^y q_1(y')\,dy' - \int_0^y N_{xy}(0,y)\,dy$$

$$= P_1(0) + \int_0^y q_1(y')\,dy' + \int_0^y \left(\frac{\partial^2 F}{\partial x \partial y}\right)_{x=0} dy$$

$$= P_1(0) + \int_0^y q_1(y')\,dy' + \int_0^y \sum_{m=0}^{M} \sum_{n=0}^{N} j_{mn} \cos\frac{n\pi y}{b}\,dy$$

$$= P_1(0) + \int_0^y q_1(y')\,dy' + \sum_{m=0}^{M} \left(y j_{m0} + \sum_{n=1}^{N} \frac{b}{n\pi} j_{mn} \sin\frac{n\pi y}{b}\right) \tag{C4}$$

Similarly

$$P_2(y) = P_2(0) - \int_0^y q_2(y')\,dy' - \sum_{m=0}^{M} (-1)^m \left(y j_{m0} + \sum_{n=1}^{N} \frac{b}{n\pi} j_{mn} \sin\frac{n\pi y}{b}\right) \tag{C5}$$

$$P_3(x) = P_3(0) + \int_0^x q_3(x')\,dx' + \sum_{n=0}^{N} \left(x j_{0n} + \sum_{m=1}^{M} \frac{a}{m\pi} j_{mn} \sin\frac{m\pi x}{a}\right) \tag{C6}$$

$$P_4(x) = P_4(0) - \int_0^x q_4(x')\,dx' - \sum_{n=0}^{N} (-1)^n \left(x j_{0n} + \sum_{m=1}^{M} \frac{a}{m\pi} j_{mn} \sin\frac{m\pi x}{a}\right) \tag{C7}$$

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(1) Green, A. E.: Double Fourier Series and Boundary Value Problems. Proc. Camb. Phil. Soc., Vol. 40, Part 3, October 1944, pp. 222-228.

(2) Scarborough, James B.: Numerical Mathematical Analysis, Fifth Edition. The Johns Hopkins Press, Baltimore, 1962, pp. 295-299.

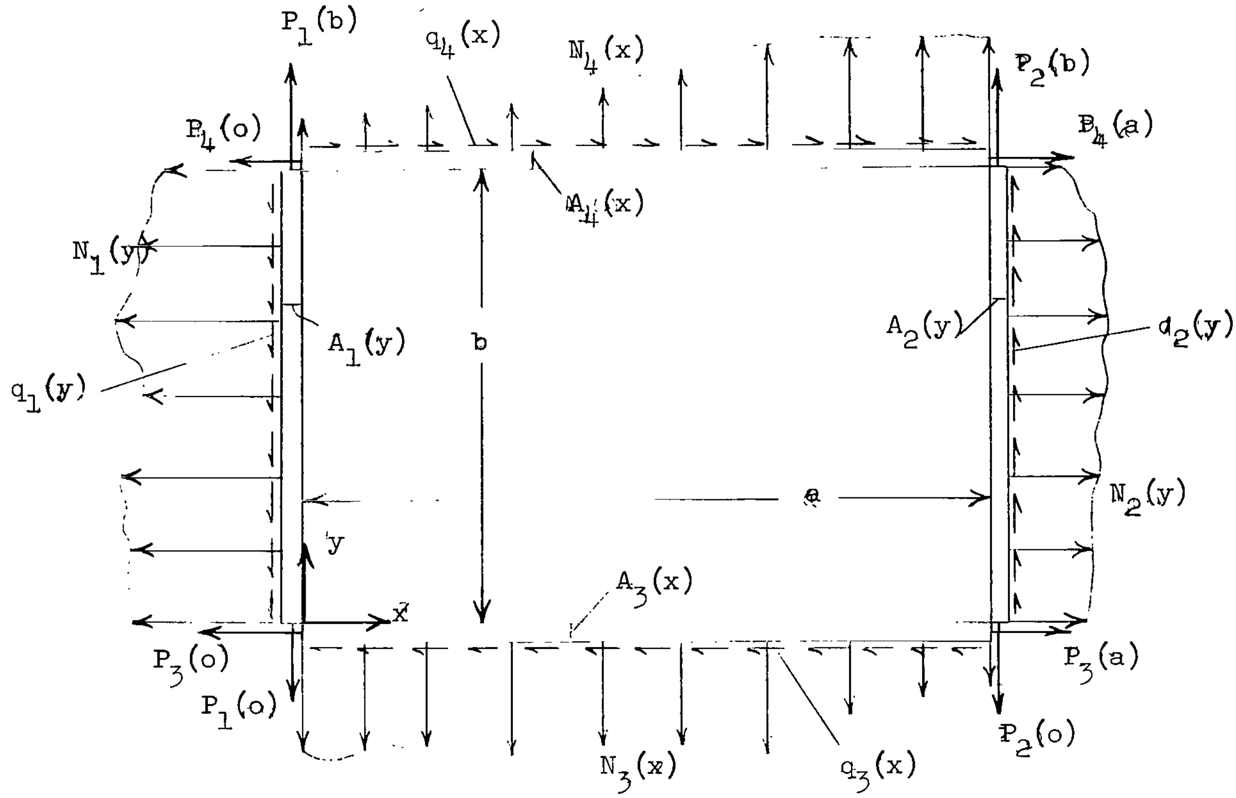


Figure 1. - Structure and loading.

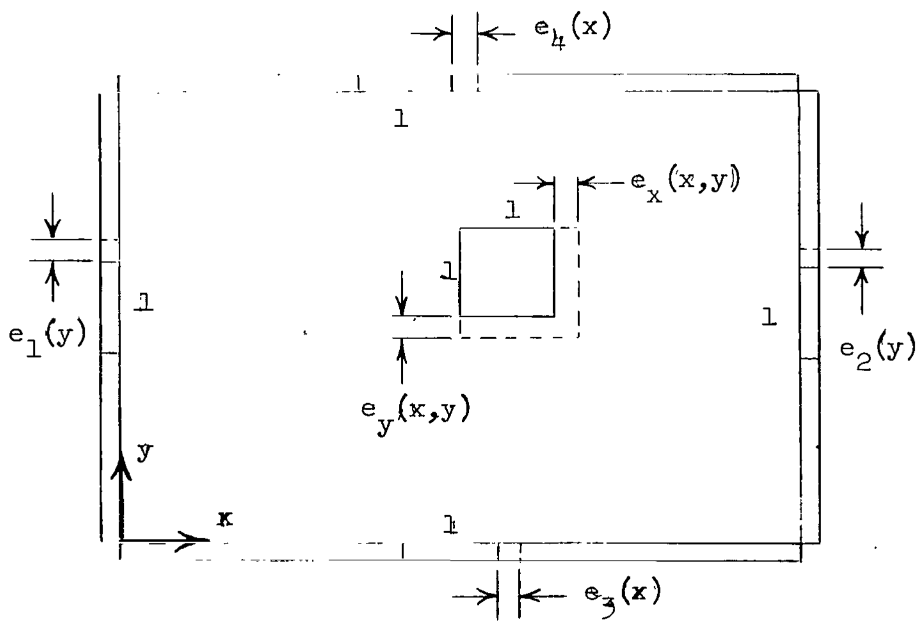


Figure 2. - Thermal strains shown schematically.

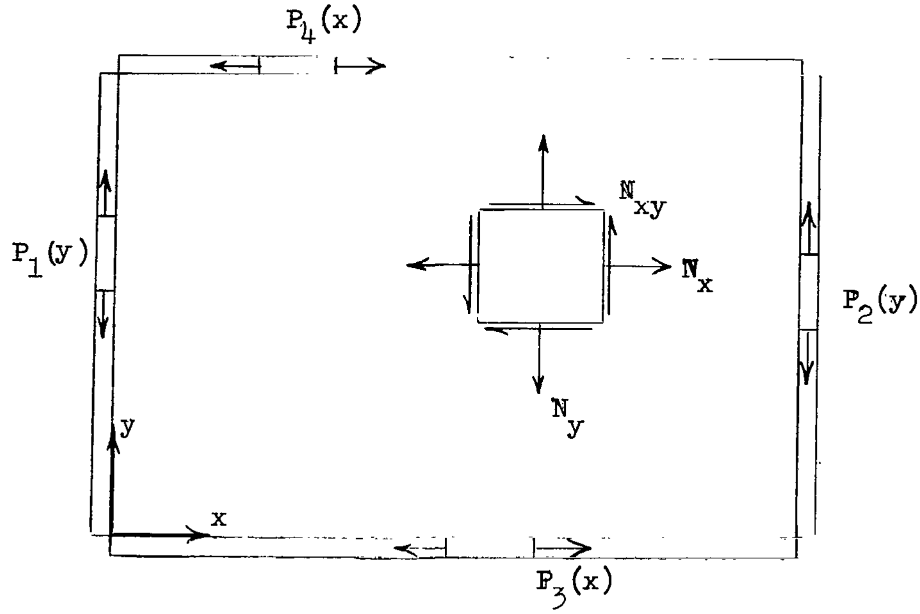
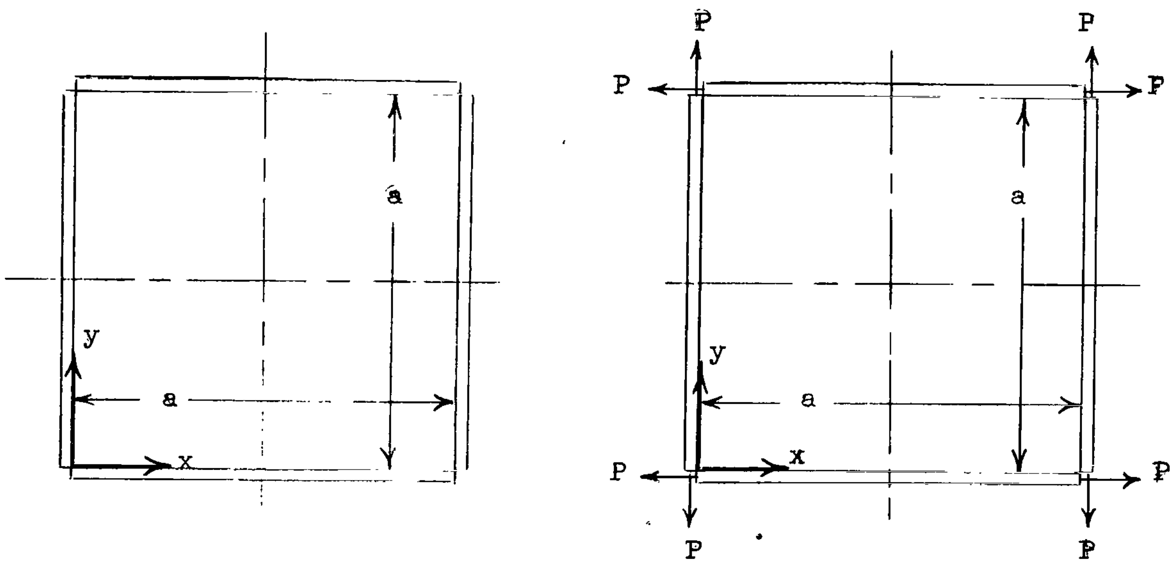


Figure 3. Notation for stiffener and plate forces.



Stiffener temp. = 0
Plate temp. = $\theta \sin \frac{\pi x}{a} \sin \frac{\pi y}{a}$
Coefficient of expansion = α

(a) Thermal-stress problem

(b) "Shear-lag" problem.

Figure 4. - Problems considered for numerical examples.

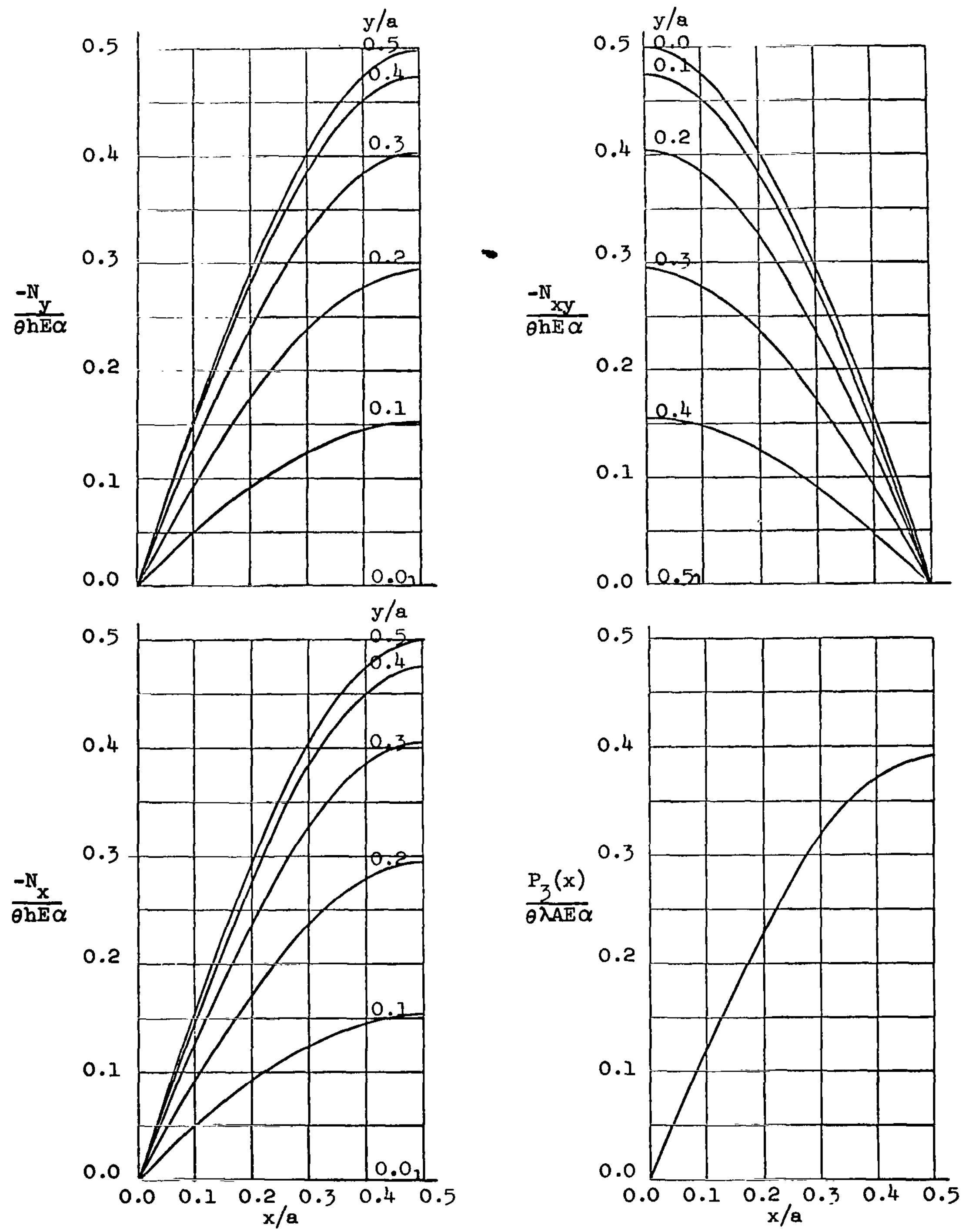


Figure 5. - Variation of dimensionless plate and stiffener forces for the thermal-stress problem. (a) $\lambda \equiv 4ah/(\pi^2 A) = 0.0$.

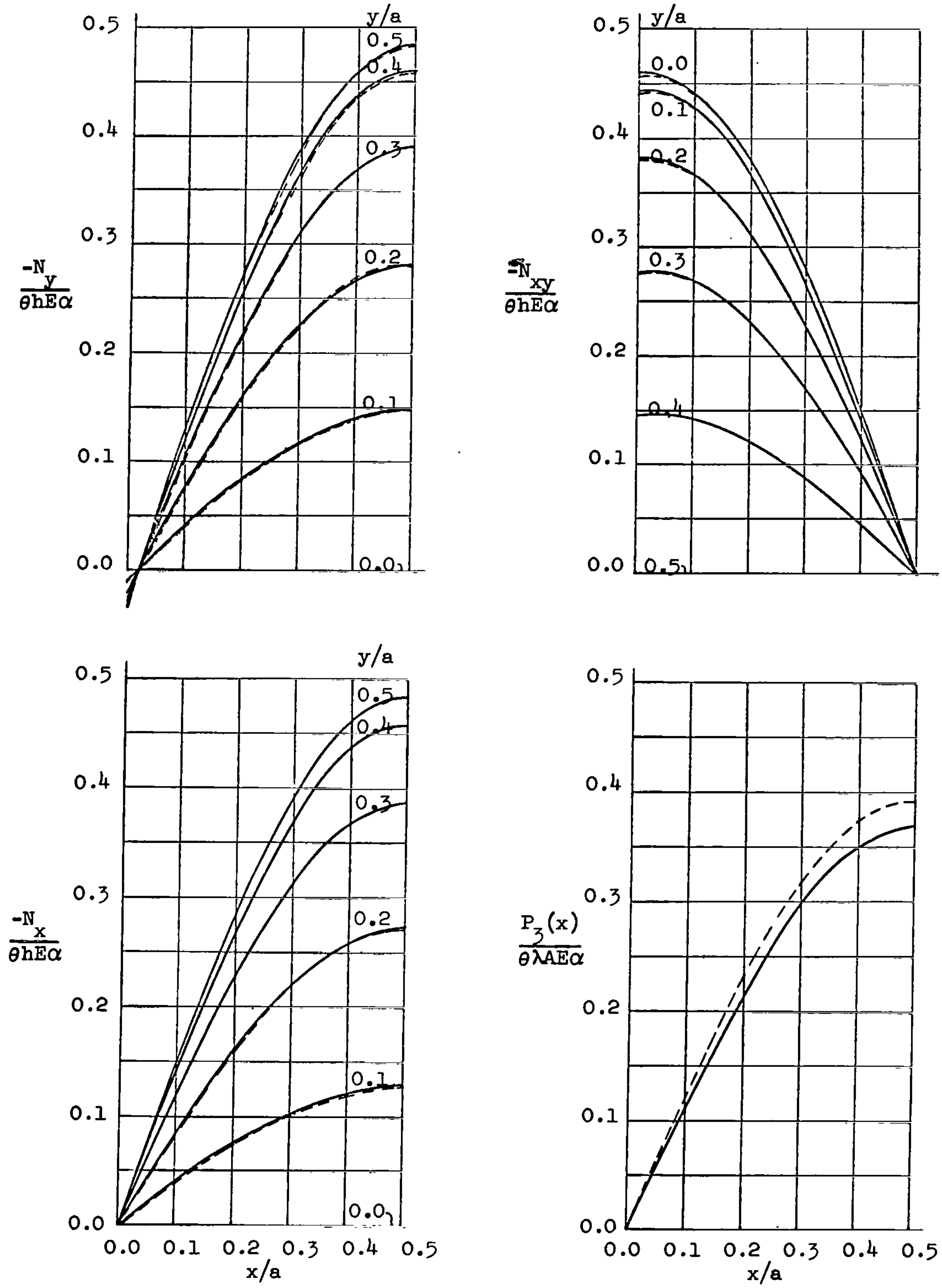


Figure 5. - Continued. (b) $\lambda \equiv 4ah/(\pi^2 A) = 0.1$; dashed curves represent approximate solution using equations (B80).

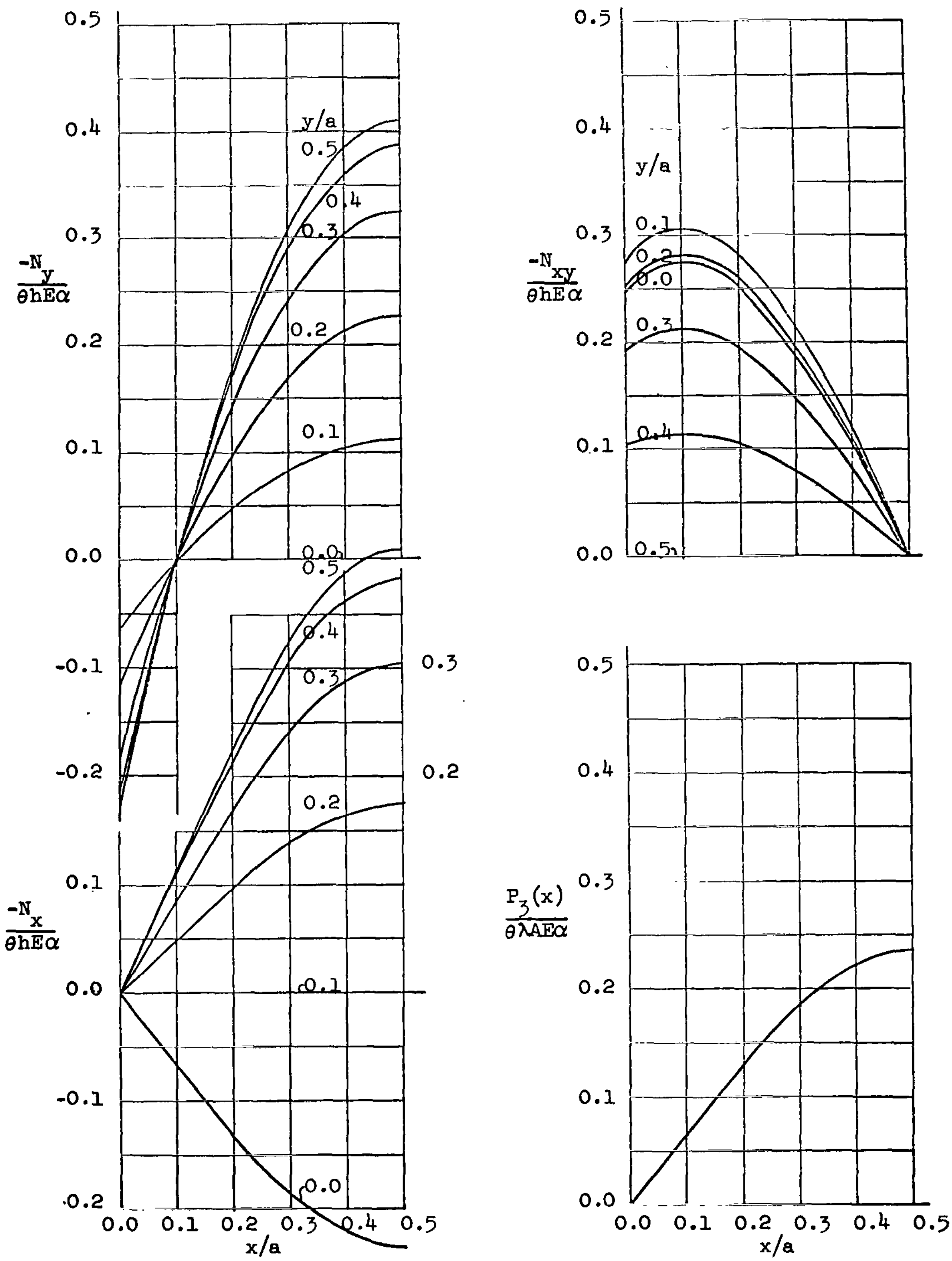


Figure 5. - Concluded. (c) $\lambda \equiv 4ah/(\pi^2 A) = 1.0$.

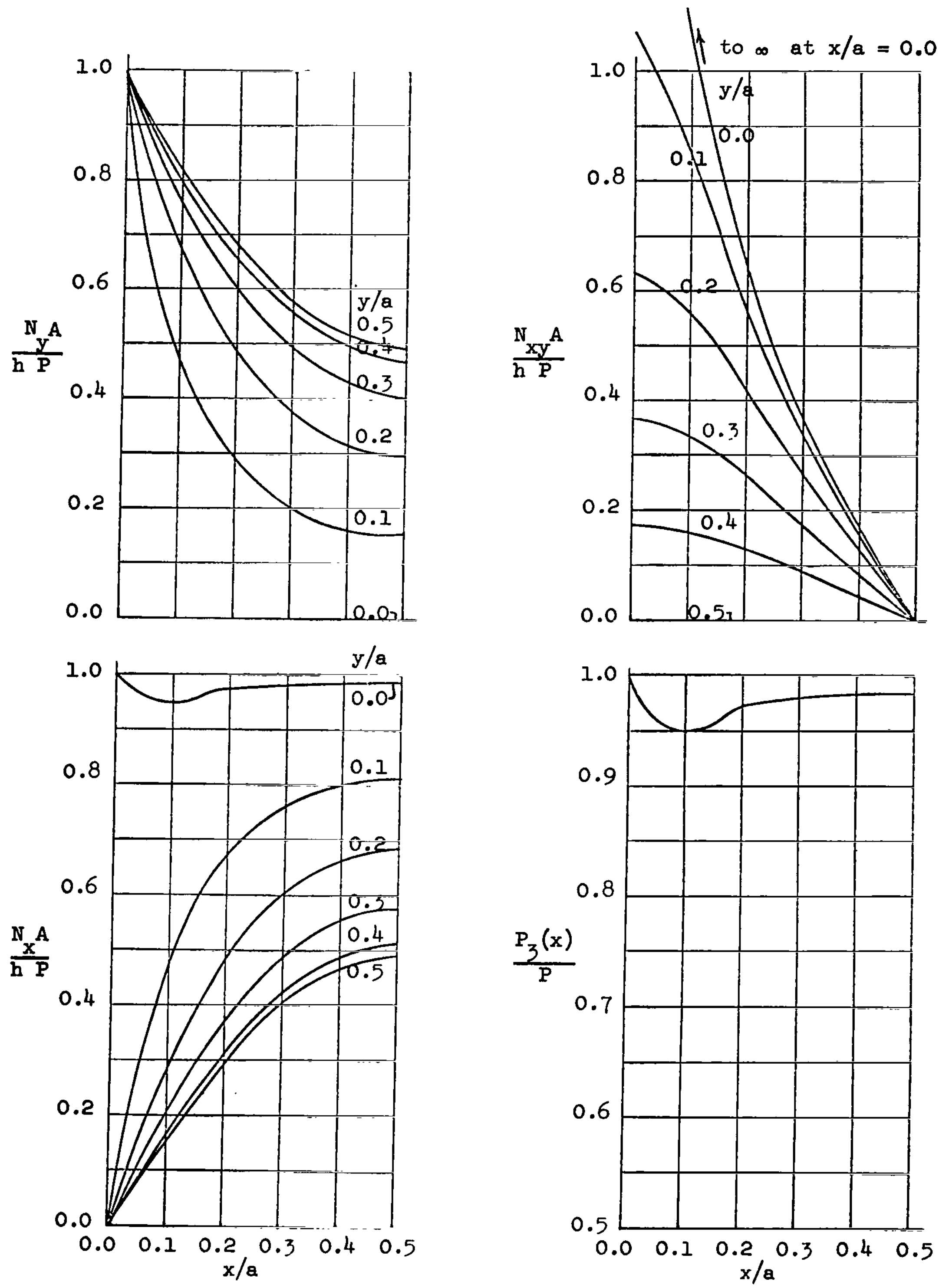


Figure 6. - Variation of dimensionless plate and stiffener forces for the shear-lag problem. (a) $\lambda \equiv 4ah/(\pi^2 A) = 0.0$.

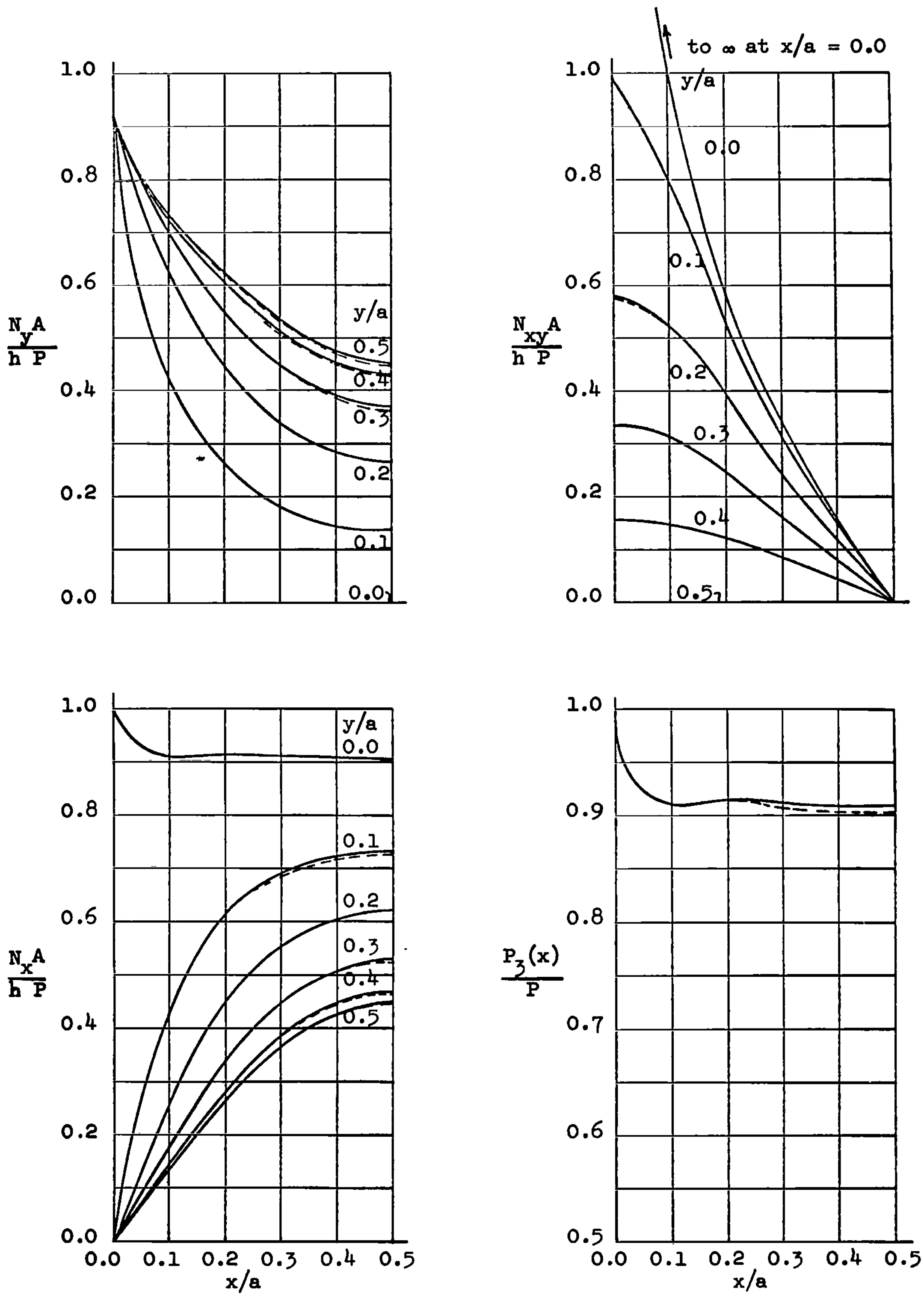


Figure 6. - Continued. (b) $\lambda \equiv 4ah/(\pi^2 A) = 0.1$; dashed curves represent approximate solution using equations (B81) to (B84).

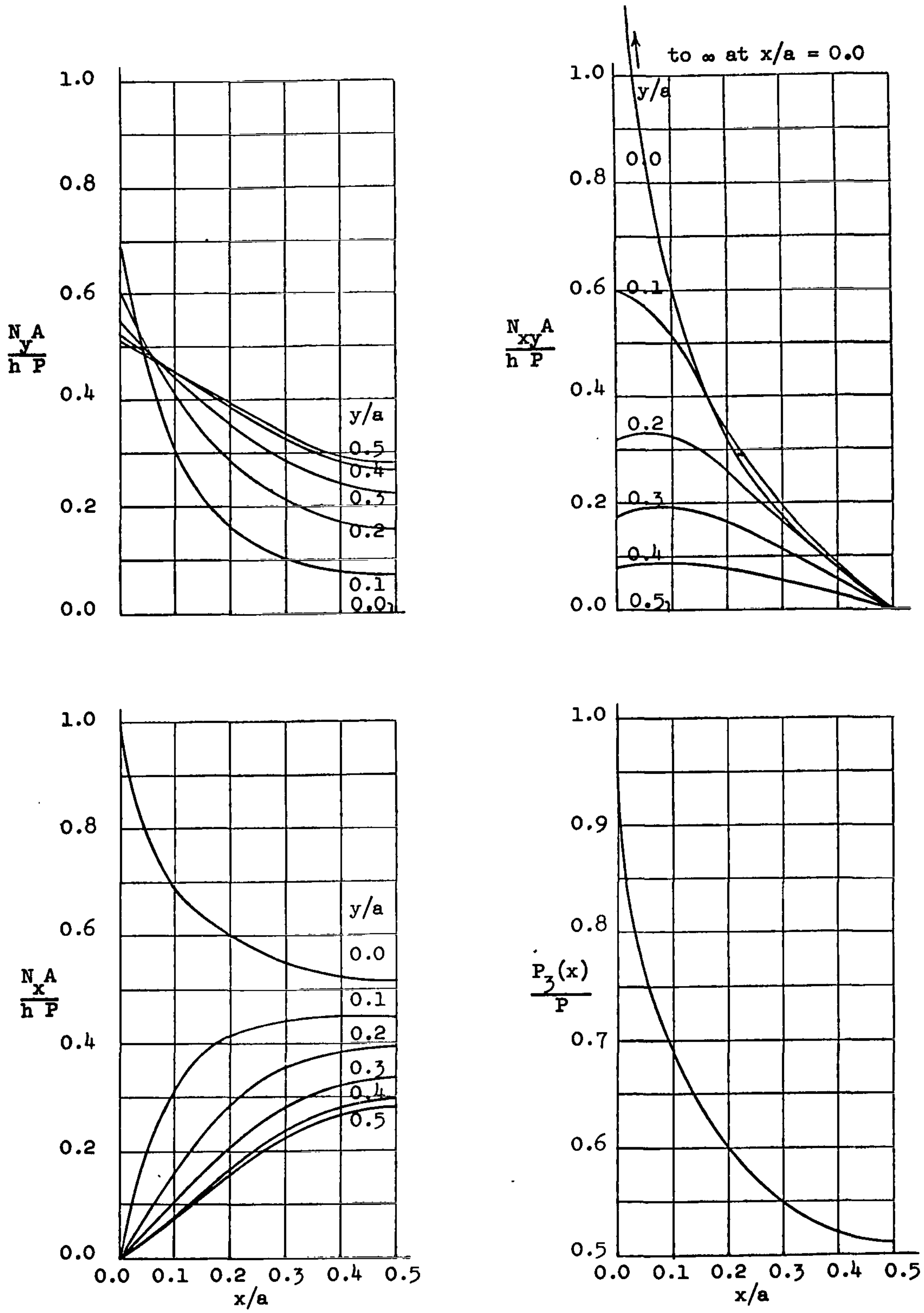


Figure 6. - Concluded. (c) $\lambda \equiv 4ah/(\pi^2 A) = 1.0$.

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